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XVII. *A Note on the Graphs of the Bessel Functions of Integral Order. By* B. HAGUE, *B.Sc., D.I.C.*

THE differential equation $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + (x^2 - n^2)y = 0$ is of very frequent occurrence in many branches of mathematical physics and it is shown in the standard treatises * that the solution of the equation is expressible in terms of special functions, known as Bessel functions. The solutions of the equation met with in the commonest practical problems are composed of functions of the first kind, defined by the infinite series,

$$J_n(x) = \sum_{r=0}^{r=\infty} \frac{(-1)^r}{\Gamma(n+r+1) \cdot \Gamma(r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r},$$

where Γ is the Eulerian gamma function, and n is generally a positive integer.

In a recent investigation the writer had occasion to plot the graph of the function for certain integral values of n, and the resulting curves seemed so interesting that a complete series of graphs was prepared for all integral values of n from zero to 12 inclusive, using the tables recently published by the British Association Committee.

In the hope that the curves may be of interest to others engaged in practical work the author submits Figs. 1, 2 and 3 for the consideration of the Society. In Fig. 1, $J_n(x)$ for $n = 0, 1, 2$, and 3 is shown; in Fig. 2 the series is extended up to $n = 7$; and in Fig. 3 is completed to $n = 12$. On the latter diagram a few points are shown of $J_{18}(x)$ in order to emphasise the striking and well-known tendency of $J_n(x)$ towards zero as n becomes large.

Apart from the purpose for which the graphs were constructed, they have also proved a useful means of verifying by graphical means the commoner properties of the functions and some of the simpler solutions of Bessel's equation.

The mathematical theory of the J functions has been investigated with the greatest thoroughness, and tables of the values of the functions have been published from time to time. However, with the exception of J_0 and J_1—the graphs of which are

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* Forsyth, "Differential Equations," and Gray and Mathews, "A Treatise on Bessel Functions."
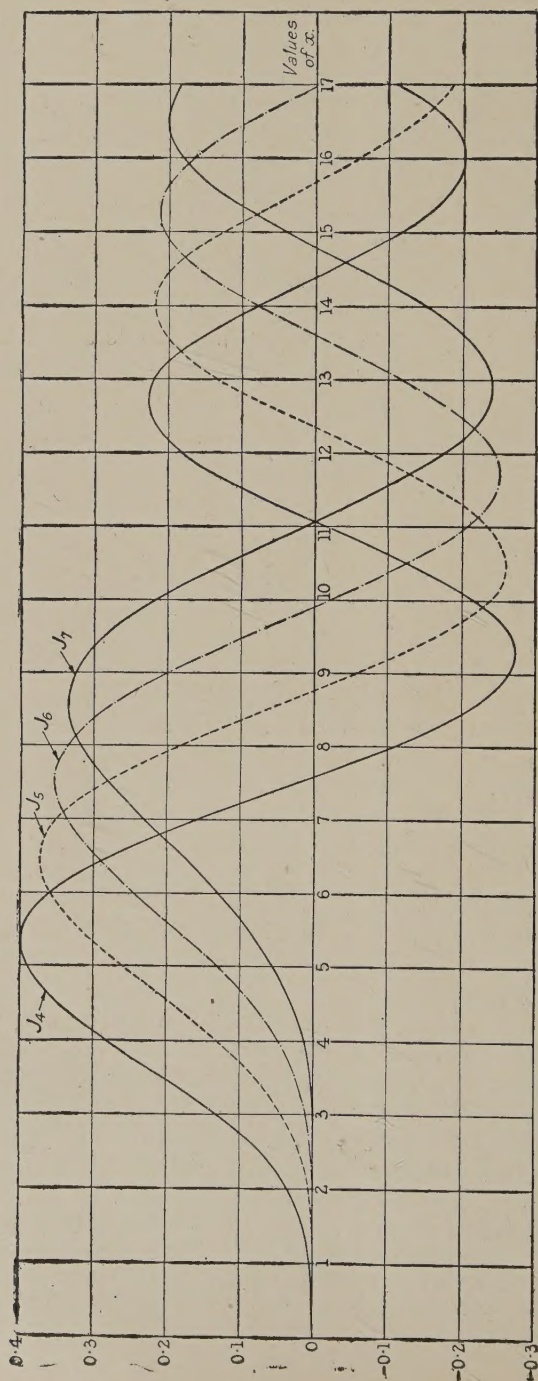
Fig. 2.—Bessel Functions, $J_n(x)$, of Integral Order. $n = 4, 5, 6, 7.$
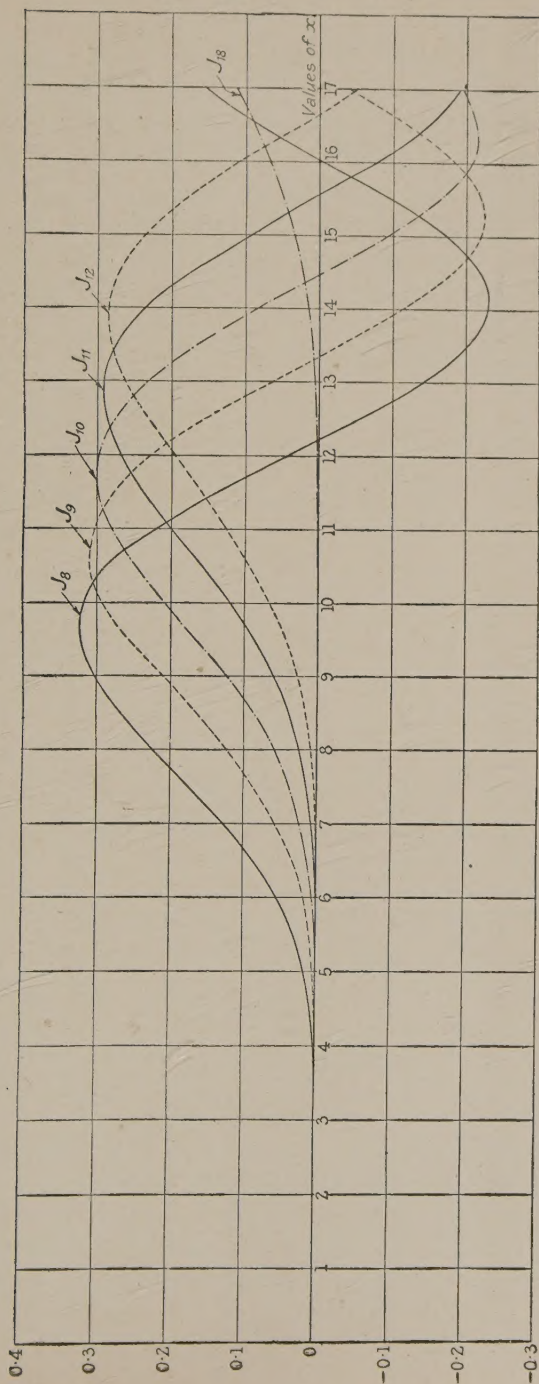
Fig. 3.—Bessel Functions, $J_n(x)$, of Integral Order. $n = 8, 9, 10, 11, 12$ & 18.

given in Gray and Mathew's treatise and of these and a few other orders in Jahnke and Emdes' " Functionentafeln "—the writer does not recollect seeing the march of the functions illustrated for integral values of the parameter n.

In concluding this short note the author wishes to thank Dr. A. R. Forsyth, F.R.S., for the interest which he has shown in the work, which was carried out at the City and Guilds (Engineering) College.

XVIII. *An Investigation of Radium Luminous Compound. By* CLIFFORD C. PATERSON, J. W. T. WALSH *and* W. F. HIGGINS.

Synopsis.

1 Introduction.

2. Measurement of Luminosity.

3. Measurement of Radium Content.

4. Measurement of Luminosity of the Markings of Luminous Dials.

5. The Best Dimensions for Luminous Markings to Secure Maximum Legibility.

6. Decay of Luminosity.

     (*a*) Luminous Compounds before Painting.

     (*b*) Painted Dials.

7. Considerations Governing the Choice and Specification of Radium Luminous Compounds.

1. Introduction.

In the present Paper is given an account of work carried out at the National Physical Laboratory on Radium Luminous Compounds. The investigations originated in a request by the Admiralty for information and assistance in connection with the use of this material for the dials of large numbers of instruments used by the Navy, and it is by permission of Capt. Creagh-Osborne, Admiralty Superintendent of Compasses, that there are included in this publication details of work wh'ch has been carried out at his request. The Compass Department of the Admiralty began the employment of radium luminous compound as far back as 1914, and was the first in the field to adopt it on an extensive scale. The use of separate lighting arrangements for instruments which require to be read at night is thus obviated.

The authors wish to say that there has been no leisure or opportunity to carry out a thorough investigation of the whole subject. Such an investigation would have included inquiry into the origin and characteristics of the various zinc sulphides used and of the effect of impurities on the luminosity of the compound, together with the effect of different methods of infusing into the zinc sulphide the very small quantities of rad:um which are used. These and similar problems have had to yield place to the necessity for fixing and applying proper

standards of luminosity both to the paint as manufactured, and to the instrument dials after the application of the luminous compound.

The standard of luminosity for the paint (generally known as the N P.L. Standard) was fixed in absolute measure in December, 1915, and that for the dials themselves shortly afterwards, and many thousands of luminous instrument dials have since been measured for conformity with the standards laid down.

The subject is a comparatively new one, for the value of radium luminous compound has only been recently recognised in connection with commercial apparatus. Photometric measurements of the brilliancy of zinc sulphide under the action of the radiations from radium products were made by Marsden in 1910,[*] who directed the radiation from a tube of emanation on to a zinc sulphide screen and measured the luminosity produced. The luminosity was found always to decay at a more rapid rate than the emanation.

In a Paper in 1915,[†] Mr. H. Hughes, describing the Creagh-Osborne compass, dealt with the use and application of radium compound to prismatic compasses, and in a further Paper to the Optical Society in June, 1916, F. Harrison Glew [‡] dealt with the general question of luminescence, and considered the practical uses to which radium luminous compound may be put, as well as the method of application of the compound to a surface in order to produce the maximum luminosity. Mr. Glew gave a further lecture on this subject to the Illuminating Engineering Society,[§] and this was followed by notes on the behaviour and testing of luminous paint by Mr. J. S. Dow and Mr. A. Blok.

A description of the apparatus used at the Bureau of Standards for measuring the luminosity of radium luminous compound has been given by N. E. Dorsey.[||]

In the present Paper the ground covered by these authors will only be touched upon in so far as is necessary for the present purpose. It may be recalled, however, that radium

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* Roy. Soc. " Proc.," LXXXIII., pp. 548-561, 1910.   " Sci. Abst.," 76 A., 1911.

† " Improvements in Prismatic Compasses," H. A. Hughes, " Trans." Opt. Soc., Lond., Vol. XVI., p. 17.

‡ " Radium and Other Luminous Compounds," F. Harrison Glew, " Trans." Opt. Soc. Lond., Vol. XVI., p. 276.

§ "Illum. Eng.," p. 72, March, 1917.

|| "Washington Acad. Sci. J.," 7, pp. 1—6, Jan. 4, 1917.

luminous compound consists almost exclusively of specially prepared zinc sulphide, with which is mixed about four parts per 10,000 of radium bromide or its equivalent in gamma ray activity.

The luminosity of the resulting powder depends on the amount of radium present and on the quality of the zinc sulphide. The reason for the superiority of one type of zinc sulphide over another is at present obscure, but the responsiveness of different grades of zinc sulphide to radioactive excitation varies very greatly and offers a wide field for investigation. The zinc sulphide responds mainly to the a-ray activity of the radium (see Glew, *loc. cit.*). As radium is bought, sold and measured by its γ-ray and not by its a-ray activity, a constant ratio is assumed between the two, and all values of radium content given in this Paper are in terms of γ-ray activity. a rays are emitted from the radium itself and from the emanation, but they are readily stopped by solids (including any organic medium used for binding). According to the so far generally recognised hypothesis the a-particles impinging on the crystals of zinc sulphide excite certain active centres. After its bombardment each active centre becomes extinct and a gradual decay of the zinc sulphide therefore results. That this theory, in its simple form, fails exactly to represent the facts is shown later, and a modification of this theory is discussed in a second theoretical Paper by one of the authors.* To apply the paint in practice the powder is mixed with a minimum possible quantity of varnish or other suitable binding material, and laid on either with a brush or by filling in the figuring which has been engraved into the surface of the object to be painted.

2. *Measurements of Luminosity.*

The measurement of the luminosity of radium paint is rendered difficult by two considerations—(i.) the green hue of the light emitted, (ii.) the very low brightness to be measured. It is desirable at the outset to make determinations of luminosity in absolute rather than relative measure, and comparisons have therefore to be made with the ordinary photometric standards.

(i.) The very green hue of the light given by radium compound necessitates the use of a green filter when photometering

* "The Theory of Decay in Radioactive Luminous Compounds," J. W. T. Walsh.

it.   Marsden examined the light spectroscopically, and found that the visual region lay between $\lambda=5,920$ and $\lambda=4.250$, with a fairly sharp edge towards the red.   It is found that Wratten Filter " Minus Red," used in conjunction with a tungsten filament lamp operating at about 1·5 watts per candle, gives an approximate colour match with the samples of rad um paint ordinarily used.   A still better match is obtained by the use of a sample of glass made by Messrs. Chance Bros., known as " B.O.T. Light Green," and corresponding with the lightest shade of green glass permitted for ships' starboard navigation lights.   The expression of the luminosity in absolute measure entails the determination of the transmission coefficient of the filter.   This has been carried out at the Laboratory by a number of observers, so that the result may correspond as nearly as possible with that which would be given by the " average " eye.
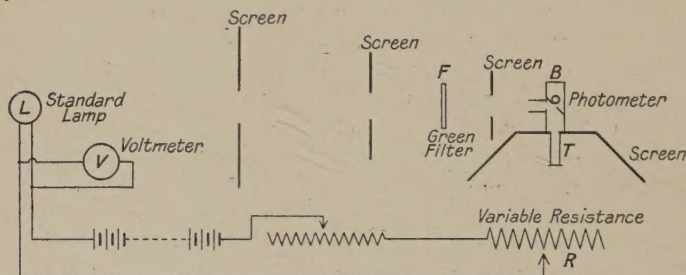
FIG. 1.—DIAGRAM OF APPARATUS FOR MEASURING THE BRIGHTNESS OF RADIUM LUMINOUS COMPOUND.

(ii.) The degree of brightness required to be measured is of the order of 10 microcandles per square centimetre, or 0·03 equivalent foot-candles.*

This is about one-hundredth of the brightness of a newspaper satisfactorily illuminated for comfortable reading by artificial light.   The brightness is of the order of that of white paper illuminated by the light of the full moon.   It will be seen that this degree of illumination is one at which the Purkinje effect is operative, so that the transmission coefficient of the green filter must be determined at an illumination corresponding with the brightness of the paint with which it is to be used,

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* This is a convenient unit of brightness, and represents a brightness equal to that of a perfectly diffusing surface with a coefficient of reflection of 100 per cent., when illuminated to an intensity of 1 ft.-candle (i.e., by a source of 1 candle placed 1 ft. away from the surface in the direction of its normal).

and it must be ensured that the surfaces to be compared are really in juxtaposition without any intervening space to separate their images on the retina.

The following description and diagram (Fig. 1) show a form of apparatus which 18 months use at the Laboratory has shown to be eminently satisfactory for this class of measurement.

L is a tungsten filament lamp mounted on the carriage of a 30 ft. long photometer bench. It is fed from a storage battery, the pressure at the lamp terminals being controlled by a resistance, R, which is to the hand of the observer. The value of the pressure applied to the lamp terminals is indicated by a voltmeter, V. F is the special green filter. It reduces the hue of the light given by L to that of the radium compound. B is a small brass box, 120 mm. long by 37 mm. square, and shown in more detailed plan in Fig. 2.*
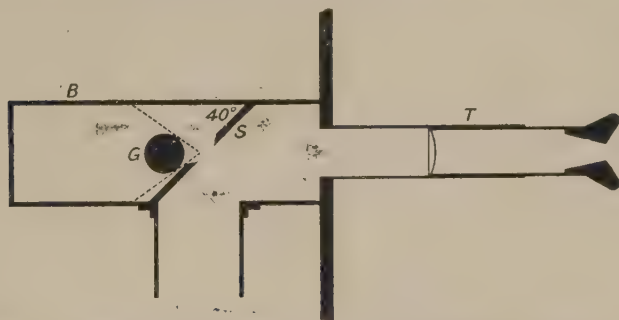
FIG. 2.—DETAIL OF BOX CONTAINING THE COMPARISON SURFACE AND TUBE OF COMPOUND.

It contains S, a sheet of matt white celluloid, set at an angle of 50 deg. to the axis, and having at its centre an oblong aperture, 4·5 mm. high by 7 mm. broad. The relative dimensions are such that when the screen is looked at through the telescope the aperture has the appearance of a small square with 4·5 mm. sides. This aperture is cut with a sharp bevel, so as to reduce to the smallest possible dimensions the black edge between the illuminated screen and the surface of the radium compound.

The radium compound is contained in a glass tube, G, and wedge-shaped guides are arranged inside the box to assist in placing it centrally behind the aperture in S, no matter what may be the diameter of the tube.

* A very similar apparatus was described by Mr. Blok to the Illuminating Engineering Society, March, 1917.

The transmission coefficient of the filter F was determined from the mean results of five experienced photometric observers, each of whom matched the brightness of three separate samples of radium paint against the brightness of the screen S. The illumination of S was varied by moving the lamp L along the line LF until a match of intensity was obtained. This was done (i) with the filter F removed, and (ii) with the filter in place. The mean ratio of the illuminations was taken to represent, for the conditions under which the apparatus is used, the transmission coefficient of the filter for an " average " eye.

The values obtained for this coefficient by the different observers differed from the mean value by $-17\cdot2$, $-6\cdot4$, $+4\cdot3$, $+8\cdot5$, and $+10\cdot8$ per cent. The mean value must not, therefore, be considered as accurate to nearer than $3\cdot5$ per cent. This inaccuracy in the absolute transmission coefficient may appear large at first sight, but a very much greater number of observers would be required to reduce it to 1 per cent.

It will be observed that the transmission coefficient was determined under conditions identical with those prevailing when the luminous compound itself is measured—$i.e.$, the size of the field and its illumination are the same—so that those errors are avoided which result from the well-known retinal peculiarities which appear in the use of green light at low illuminations.* The mean coefficient obtained was $2\cdot15$ per cent. This value was confirmed by means of a portable photometer, the illumination of a white card being measured directly (a) without the filter, (b) with the filter interposed.

The other constant to be determined was the coefficient of reflection of the white card S as viewed from the eyepiece T, with the light incident from L at an angle of 40 deg. This was determined by measuring the brightness of the card as viewed from the direction of T with a portable photometer, and comparing this with the brightness of another portion of the same piece of card when held normal to the direction of the light.

The coefficient obtained in this way embodied both the coefficient of reflection K_θ of the card with the light incident at an angle θ, and the reduction of illumination according to the cosine law. This combined coefficient $K_\theta \cos \theta$ was found to be $68\cdot2$ per cent.

To obtain now the brightness of a sample of paint T from the

* "Colour Phenomena in Photometry," J. S. Dow, " Proc." Phys. Soc., Vol. 20, 1906, p. 245.

candle-power C of the lamp L at distance d when the brightnesses of S and T are equal, we have :—

$$B = \frac{C}{d^2} \times 0.21_5 \times 0.68_2.$$

There is, however, one more point yet to be considered. It will be clear that the transmission coefficient of the filter F will vary with the hue of the light traversing it, so that the value of the coefficient found above holds only for the light from a tungsten filament lamp operating at the one definite efficiency, at which the determination of its transmission coefficient was made. The result of this is that variation of illumination on S has to be obtained by change of position of L along the line LS rather than by change of voltage L. In actual use, however, when measuring the brightness of a sample of radium compound, it is found most convenient to fix L in such a position that an approximate match is obtained in S with a voltage on L of 95. Several settings are then made by means of the resistance R and as the extreme voltage variation on L thus obtained does not generally exceed 2 volts, it is safe to assume that over this range the variation in the hue of the light from L will not appreciably affect the value of the transmission coefficient of F. The candle-power variation with voltage of L has been determined accurately on the photometer bench, over the range from 92 to 98 volts, and from the voltage indicated by V, the distance LS, and the formula given above, the brightness of the paint in T is at once found.

Absorption of Glass Tubes.—The radium paint being in powder form, it is necessary to measure its brightness when enclosed in a glass tube. This is desirable for two reasons. Firstly, it is necessary to obtain a smooth surface of the powder in order that it shall be of uniform brightness, because any cavity in the surface shows a greatly enhanced brilliancy due to internal reflection. Secondly, it is necessary to avoid any possible loss of emanation.

From a series of experiments made when the apparatus was first set up, it was found that variations in the thicknesses of the glass containing tubes, up to at least 1.3 mm. makes no difference to the brightness of the paint viewed through them.

The colouration of the glass by the action of the radium, on the other hand, produces a progressively increasing absorption for which allowance has to be made in considering the

decay of luminosity of the compound.* Lead glass gives a brown coloration and soda glass a purple one.

Measurements have been made of the amount of this effect in the case of six of the samples of compound studied, all of which were contained in soda glass tubes. The absorption was found by making alternate measurements of the luminosity of a sample of compound when contained (a) in the vessel under examination, (b) in a new vessel not previously exposed to γ-radiation. The results obtained are tabulated below and the value of luminosity given throughout the Paper are corrected on the assumption that the light absorption has followed a linear law of increase. As the whole correction does not in any case exceed 16 per cent. this assumption will not lead to any appreciable error in the final corrected values of luminosity.

TABLE I.

Sample.	Age of compound.	Total absorption due to coloration.
3	14½ months.	13 per cent.
6	17 ,,	8 ,,
7	17 ,,	9 ,,
8	11½ ,,	10 ,,
9	13 ,,	10 ,,
10	13 ,,	16 ,,

3. *Measurement of Radium Content.*

The general method of determination of the quantity of radium contained in a concentrated sample is by direct comparison with a standard consisting of a known amount of one of the salts of the element. The means by which this comparison is carried out is based on the ionisation produced by the γ-rays emitted from the samples considered.

Radium gives out three types of rays—namely, the α, β and γ-rays—and the ionisation produced by any of these could be made use of as a measure of the quantity of radium present. Two difficulties, however, arise in attempting to use either of the two former types of radiation. Firstly, in the case of radium in radioactive equilibrium, it is not possible to isolate these rays, so that the properties of the α or β-rays alone are employed. The same trouble does not arise in the case of γ-rays as these are very penetrating and may consequently be freed from α or β-rays by the use of suitable screens of absorb-

* Glew, *loc. cit.*, p. 287.

ing material. α-rays are absorbed by about 8 cm. of air at atmospheric pressure or by very thin layers of solid matter, while the β-rays are completely stopped by about 1 cm. of lead. In practice it is found that a sheet of lead 5 mm. thick reduces the proportion of β-rays present to a negligible amount compared with the γ-rays. Consequently in the apparatus to be described later the ionisation chamber is covered by a lead disc 5 mm. thick, and the ionisation produced inside this space must therefore be solely due to the γ-rays. In the second place the intensity of all the rays changes with time until complete radioactive equilibrium is attained. Consequently the α or β-ray activity of any sample of radium and its radioactive products will not be sufficiently constant for measurement purposes for some years. For the γ-rays, however, a constant intensity is reached in a comparatively short time (about 30 days suffices) as the products of disintegration beyond radium do not emit γ-rays (with the exception of some very soft rays, which in practice are completely absorbed before the ionisation chamber is reached). For purposes of measurement it is necessary to prevent the escape of the gaseous product— viz., radium emanation—and to this end the material must be sealed up in glass or metal tubes for a sufficient length of time before comparisons can be undertaken.

The γ-ray method is primarily intended for the estimation of high concentration, but may also be extended to estimation of the total quantity of radium in a sample of radium luminous compound if the necessary precautions as detailed later are taken.

The smallest equivalent quantity of radium bromide which can in this way be accurately measured is about 1 mgm., and the quantity of radium luminous compound on which the test is made should hence be from 3 to 5 grms.

The standard against which the samples of radium luminous compound are compared at the National Physical Laboratory, consists of a preparation of pure radium bromide, equivalent to 5·20 mgm. of hydrated radium bromide ($RaBr_2 2H_2O$). This is compared at intervals with the British Radium Standard, which was certified by the International Radium Standards Committee after comparison with the International Standard at Sèvres.

The apparatus employed is represented in Fig 3. It consists of a massive wooden bench along which a stand, A, can be moved. The position of this stand with respect to the

ionisation chamber is indicated by a pointer and scale.   The
samples under test are placed in turn in the V-support on
the stand A, and this is so arranged as to bring the sample
approximately to the level of the centre of the ionisation
chamber.

The ionisation chamber C consists of a lead cylinder of
internal diameter 140 mm. and depth 94 mm.   The thickness
of the walls is 6 mm.   The front of this chamber consists of a
lead plate of 5 mm. thickness ; this is held in close contact
with the walls of the chamber by suitable clamping screws.
Alternative lead plates of different thicknesses are available as
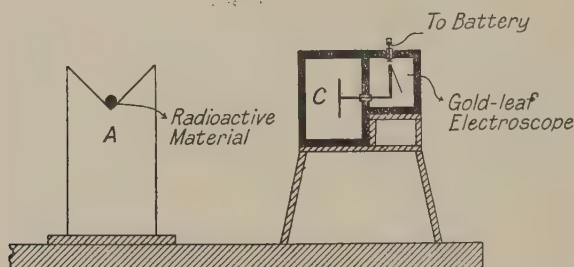desired.   The central insulated disc is supported from the back

FIG. 3.—DIAGRAM OF APPARATUS FOR MEASURING RADIUM CONTENT BY
THE γ-RAY METHOD.

of the chamber by means of a brass wire passing through an
amberite* bush.

The gold leaf electroscope by which the ionisation current is
measured is attached to the ionisation chamber.   The gold
leaf is fastened on to a vertical projection from the brass sup-
porting wire of the disc above referred to, and is surrounded by
a square built up box of thick lead plates.   Windows are pro-
vided at each side of this box by which the gold leaf may be
illuminated and viewed.   These windows are of lead glass, and
are screened by lead shields to prevent as far as possible the
entrance of stray and scattered γ-radiation.   The insulated
system, consisting of the disc in the ionisation chamber, the
gold leaf and the support are positively charged by means of a

* This material is composed of highly compressed amber chips or dust, and
provides an excellent insulator, which may be turned to any desired shape.
Care must be exercised in doing this as the material is somewhat brittle.   It
is not hygroscopic, nor does it show any appreciable soakage effect.

charging key carried on the top of the electroscope. On pressing down the plunger of this, contact is made with a battery of accumulators of the required voltage (about 200 volts). The case and all metal parts of the apparatus other than the central insulated system are earthed, as is also the negative pole of the battery.

The motion of the gold leaf is viewed through a reading microscope magnifying about 10 diameters, and provided with a scale in the eyepiece. This instrument is held rigidly by means of an adjustable stand, and when once put into position is not moved throughout the complete series of readings.

The readings are always taken over a definite portion of the eyepiece scale, the time being determined by means of a stop-watch for the gold leaf to move over this distance. The procedure generally adopted in making a comparison is as follows : The natural leak of the system is first determined. This is done by charging up the gold leaf and noting the time taken by it in moving over the measured distance, all radium being removed from the neighbourhood of the measuring instruments. The radium standard is then placed on the stand A, which is moved to some definite position along the test bench according to the strength of the sample to be measured. For small quantities the stand is brought relatively near the ionisation chamber, while for larger amounts the distance between the two is increased. The rate of leak is then deter-mined for the standard sample, the mean of a number of observations being taken. The standard is then replaced by the material under test without the relative positions of any parts of the apparatus being changed and the rate of leak is again determined. Further readings of the standard are then taken, which should agree with those first obtained.

The comparison is then repeated for other distances between the stand A and the ionisation chamber C. Finally, the natural leak is taken at the end of the series of observations. It must be noted that all radium preparations except that actually on the test bench, must be removed as far as con-venient from the γ-ray apparatus, otherwise the natural leak will be large, and the corresponding accuracy of the compari-sons will be greatly reduced.

The correction for the natural leak is made thus : If T_0 is the time taken for the leaf to move over n divisions when the sample of radium is under observation and T_n the number when it is

absent (*i.e.*, the natural leak) then the corrected value of the time (T) will be given by—

$$\frac{n}{T} = \frac{n}{T_0} - \frac{n}{T_n},$$

i.e.,

$$T = \frac{T_0 T_n}{T_n - T_0}.$$

The corrected value of the time is thus obtained for the sample under test and for the standard. The ratio of these times then gives the ratio of the quantities of radium present in the standard and the sample under test, thus :—

$$\frac{\text{Weight of radium in sample}}{\text{Weight of radium in standard}} = \frac{\text{Time for standard.}}{\text{Time for sample.}}$$

Correction for Absorption.—From the above description it will be clear that in determining the radium content of a sample of luminous compound by measurement of its γ-ray activity, allowance must be made for the absorption of the γ-rays by the zinc sulphide. If, first, we suppose that the whole of the radium salt is concentrated along the axis of a cylindrical vessel containing the compound, it is known that the intensity of the γ-ray activity in any direction perpendicular to the axis (and at a distance from it sufficient to make the length of the vessel negligible in comparison with this distance) will be given by $I' = Ie^{-\lambda a}$, where I is the intensity which would be found in the absence of the zinc sulphide, a is the radius of the vessel, and λ is the absorption coefficient of the zinc sulphide.

In actual practice, however, the radium salt is not concentrated at the axis, but is uniformly distributed over the cross-section of the cylinder, so that the above formula does not accurately apply, although *a priori* reasoning from the known value of λ shows that for vessels of small diameter this formula is sufficiently accurate for all practical purposes. When the diameter reaches 20 mm. or 30 mm., however, it becomes necessary to investigate the effect of uniform distribution more thoroughly, and this may be done in one of two ways—(*a*) approximately, by first treating the simple case of a vessel of square section and then regarding the cylindrical vessel under consideration as made up of any convenient number of equal square-section vessels, laid alongside one another ; or (*b*)

rigidly by obtaining a convergent series with constant coefficients which will give at once the required value of intensity, from a knowledge of the value of λ. Methods (a) and (b) are described in greater detail in Appendix A

The value of λ has been determined experimentally by mixing a measured quantity of radium salt with an amount of zinc sulphide sufficient to produce radium luminous compound of standard specification, and then determining the γ-ray activity of the resulting mixture. The contents of a tube of radium salt equivalent to 11·4 mgm. of hydrated radium bromide, as determined by a γ-ray test were m xed, without loss, with 28·5 grms. of active zinc sulphide, and the mixture was enclosed in a glass bottle of known dimensions. The γ-ray activity was measured at intervals until a constant value was obtained. This was equivalent to 8·2 mgm. of hydrated radium bromide, and from this result a value for λ is calculated from the formula on page 226. This value is $\lambda = 0·03 \text{ mm}^{-1}$.

A confirmatory test was made by the use of a second sample consisting of 30 grams of radium luminous compound, which was measured in bulk, and then was subdivided into a series of six smaller quantities enclosed in glass tubes of various diameters.

The γ-ray activity of each of these was determined and was corrected for absorption in accordance with the value obtained in the first test. The radium content per grm. was found to be the same, to within an accuracy of 2 per cent., for each tube and for the sample in bulk when this had been done.

An attempt was also made to obtain the absorption coefficient by surrounding a sealed tube of radium bromide with a known thickness of zinc sulphide as mentioned above. The value so obtained leads to a result of $\lambda = 0·0092_5 \text{ mm}^{-1}$, which is roughly one-third of the value obtained in the case of uniform distribution. No explanation of this discrepancy is at present available, but further work is proposed to investigate the matter. In the meantime the first value is assumed to hold, as the conditions under which it is obtained most nearly represent the conditions of actual test of samples, and the table following gives the factors by which the measured radium content must be multiplied in order to give the true content for different sizes of containing vessels.

TABLE II.

$(\lambda = 0 \cdot 03 \ mm^{-1})$.

Diameter of containing vessel.	True content / Measured content.
5 mm.	1·06
10 mm.	1·13
15 mm.	1·20
20 mm.	1·28
25 mm.	1·35
30 mm.	1·43

4. *Measurement of Luminosity of the Markings of Luminous Dials.*

The chief commercial use of radium luminous compound lies in its application to instrument dials for rendering the figures and graduations on these visible at night without external illumination. The method by which this application may be made has been described by F. H. Glew (*loc. cit.*), and it was early found that the various methods employed by different makers of the dials gave very widely differing results. The luminosity of the figures, therefore, depends not only on the quality of the radium luminous compound employed, but also on the method of its application to the dial, so that the employment of a good quality of luminous compound is not a necessary guarantee of a satisfactory dial.

It becomes necessary then, to test the actual brilliancy of the luminous markings, and the only satisfactory method of doing this is to compare the dial under test with another dial, the markings of which are of the same pattern and dimensions, and of a known brightness. Hence the first test employed was comparison with a radium painted dial of what was considered to be good luminosity. This was very soon abandoned, however, in favour of a more convenient and fundamental test. In the first place the possibility of a deterioration of luminosity was early recognised as one of the dangers of a radium-painted standard dial, while the large number of d fferent patterns of dial to be tested would have necessitated a considerable collection of radium painted standards. These would necessarily have been at variance as regards brilliancy, since absolute uniformity is practically impossible of attainment, especially under the condit ons prevailing at the time when the standards were first set up. A still more fatal drawback was the fact that beyond the statement that the test dial did or did not equal in brilliancy the standard dial, no value

could be assigned to the luminosity of its markings to serve as a
guide for future work by the manufacturer.

In view of all these considerations a method was devised
which has from the very first given complete satisfaction in
working. This method is, in essence, the comparison of the
test dial with a metal stencil cut to the same pattern and dimen-
sions as the luminous markings of the dial, and having behind it
a screen illuminated by light of the same hue as that given by
the radium compound. The intensity of illumination of this
screen is variable by means of a resistance in series with an
electric lamp, and thus a knowledge of the variation of illu-
mination of the screen with the current taken by the lamp
gives at once the value of the brilliancy of the markings on the
test dial. The actual apparatus is shown diagrammatically in
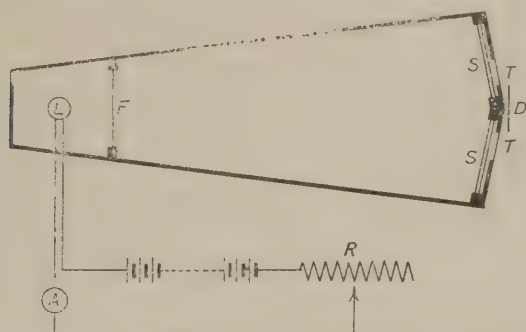Fig. 4, where a box encloses at one end an electric lamp, L, in

Fig. 4.—Diagram of Apparatus for Measuring the Brightness of
the Markings of Luminous Dials.

circuit with a resistance, R, and precision ammeter A. R is to
the hand of the observer. L is actually a 105-volt 60-watt
metallic filament lamp operated, when giving standard bril-
liancy, at about 5 watts per candle. In front of L is a sheet of
Wratten "Minus Red" filter F, in order that the hue of the
light may match that of the radium compound The green
light then falls on two screens S, S, each consisting of two thick-
nesses of thin white paper with a space of 10 mm. between
them. The two thicknesses of paper were found necessary
to secure proper diffusion of the light and it was also found
desirable to whiten the inside of the wooden strips separating
the two layers of paper in order to avoid any decrease of
luminosity at the edges. In front of the screens are placed
two stencils T, T, each consisting of a sheet of thin brass or
copper 0·3 mm. thick, framed in wood to fit the dimensions of

                                                        T 2

the screens (255×275 mm.).   The figures and graduations are cut in brass on an engraving machine, so as to be identical with the luminous markings of the dials under examination. Between the stencils is fixed the dial under test D.   It was originally intended that the test dial should be placed above the stencils so that the three dials under comparison should be one at each apex of an equilateral triangle.   This arrangement was abandoned, however, as it was found during the course of the initial experiments that different luminosities were assigned to a dial according as it was placed above, below, to the left or to the right of the dial with which it was being compared.   This effect is apparently one of personal error depending on the distribution of sensitivity to green light over the retina of the observer.   In different observers the effect was found to vary in amount, and also to change sign.   In these circumstances it was found best to give the dial an intermediate position between two comparison dials of equal luminosity.   In making
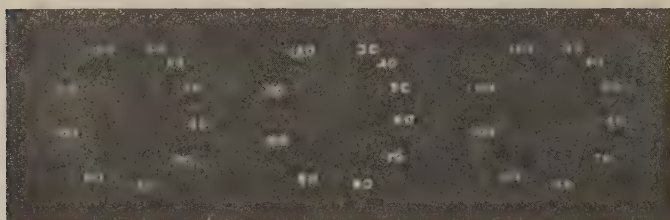

Fig. 5.—Photograph showing the Appearance of a Luminous Dial and its Comparison Stencils in the Dial-testing Apparatus.

a test of a luminous painted dial by this method the dial, as has been said, is placed between the two stencils of similar pattern, while the operator, seated with his eye on a level with and about 3 ft. from the dial varies the brightness of the comparison stencils by means of the resistance R.   As soon as he is satisfied that the three dials are of equal luminosity he notes the current indicated by the ammeter A, and from a table finds the percentage of standard luminosity of the dial under test.

This test is, of course, performed in a dark room, and it is necessary for the observer to remain sufficiently long in the dark for his eyes to become " dark-adapted."   The appearance of the test dial and its corresponding stencils is shown in the photograph (Fig. 5).   It may be here noted that the luminosity of the markings on standard dials is 0·0075 equivalent

foot-candles (2·7 microcandles per square centimetre), and that to obtain the above photograph with an Imperial Standard rapid plate an exposure of $2\frac{1}{2}$ hours was required.

An examination of the negative shows that, although there was a luminosity match between the three dials, the light from the centre one, painted with luminous compound, was more actinic.

5. *Investigation of the Best Dimensions for Luminous Markings to Secure Maximum Legibility.*

In addition to the question of luminosity of the markings of a radium painted dial, there is also the consideration of legibility, which will depend on the form and size of the markings as well as on their luminosity. An investigation was undertaken at the request of the Admiralty to determine the nfluence of these factors.

It is obvious that if the luminous lines or markings of a card are reduced in breadth too much, the letters will become illegible, because the total amount of light has become so small.
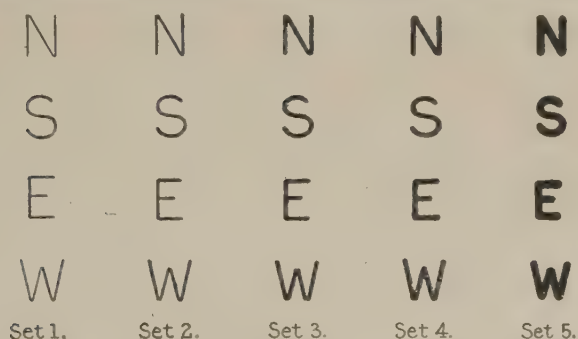

| Set 1. | Set 2. | Set 3. | Set 4. | Set 5. |

FIG. 6.—THE TYPES OF FIGURES USED TO INVESTIGATE THE RELATION BETWEEN DIMENSIONS AND LEGIBILITY.

Conversely, if the lines are increased in width in width beyond a certain point, in spite of the large amount of light emitted, the letters will lose legibility and appear merely as blotches of light. Between these two extremes there must lie a region where the legibility is a maximum. To determine this optimum width of line, observations were made on a series of letter stencils in which the letters were all 16 mm. high, but in which the width of line varied from 0·5 mm. to 3·0 mm. The letters are shown in Fig. 6. The first set with the thinnest line was used as a standard and the other sets were compared

with it. The specific luminosity of each set of letters was independently variable, and was so adjusted that the *total* light emitted was the same for each set. This adjustment was made by reducing the brightness of the broader letter by an appropriate number of thicknesses of fine black tulle. Thus, any two sets of letters when compared from a distance appeared equally bright, and it is of interest to note that under these conditions the difference in the width of the lines could not be detected.

If, however, nstead of comparing the letter from the point of view of brightness, the observer made a judgment of the relative legibility, a difference was in some cases noticeable. The set having the broadest lines (set 5) compared with the thinnest (set 1) was clearly less legible. The falling off in legibility was not, however, so noticeable between sets 1 and 4, and was inappreciable between sets 1 and 3.

In order to get a quantitative measurement of the falling off of legibility of sets 5 and 4, their *luminosity* was increased until the *legibility* equalled that of set 1. Equality resulted when set 5 was increased about 40 per cent. and set 4 about 20 per cent. The results are summarised as follows :—

TABLE III.

—	Width of line.	Ratio of width of line to height of letter.	Legibility.
Set 1	0·5 mm.	1/32	⎫ All equally legible for
Set 2	1·0 mm.	1/16	⎬ equal total luminosity.
Set 3	1·5 mm.	1/11	⎭ nosity.
Set 4	2·0 mm.	1/8	Fall of legibility =20 p.c.
Set 5	3·0 mm.	1/6	Fall of legibility =40 p.c.

It will thus be seen that for a certain specific luminosity, set 5 will not appear very much more legible than set 3, because although set 5 is twice as bright, it suffers a falling off in legibility of about 40 per cent. An experimental trial was made to test these deductions and was found to confirm the results stated.

It was concluded from these experiments that the proportion of width of line to height of letter for maximum legibility should be about 1 to 8 (set 4), and should not much exceed 1 to 7, at any rate for letters and figures in which the height exceeds about 5 mm. Below this limit experience suggests that the breadth of line may be usefully increased to one-sixth of the height of the figure.

6. *Decay of Luminosity.*

Glew points out * that the luminosity of radium compounds falls off with time. The radium itself decays to half its value in about 2,000 years, and it has sometimes been assumed that radium paint will retain its luminosity for a similar period. This is not, of course, the case, as Marsden showed in his experiments on zinc sulphide, willemite and barium platinocyanide. His source of radiation, however, being radium emanation, was itself decaying at a comparatively rapid rate,

Fig. 7.—Diagram showing the Variation throughout Life of the Luminosity of Sample 3 of Radium Luminous Compound. Inset, the Early Portion of the Life on an Enlarged Scale.

while the intensity of the radiation far exceeded anything practicable in the preparation of ordinary commercial radium compounds.

The authors have made systematic tests to determine as accurately as possible the loss of brilliancy of the radium compound ordinarily used, both in powder form, before painting,

* *Loc. cit.*, p. 216.

and also after mixture with the varnish used in its application to the surface to be treated.

(a) *Luminous Compounds Before Painting.*—Nine samples from several makers, and containing different proportions of radium salt mixed with various qualities of zinc sulphide have been retained under observation at the National Physical Laboratory. In some cases it has been possible to obtain measurements of the brightness at intervals over a period of
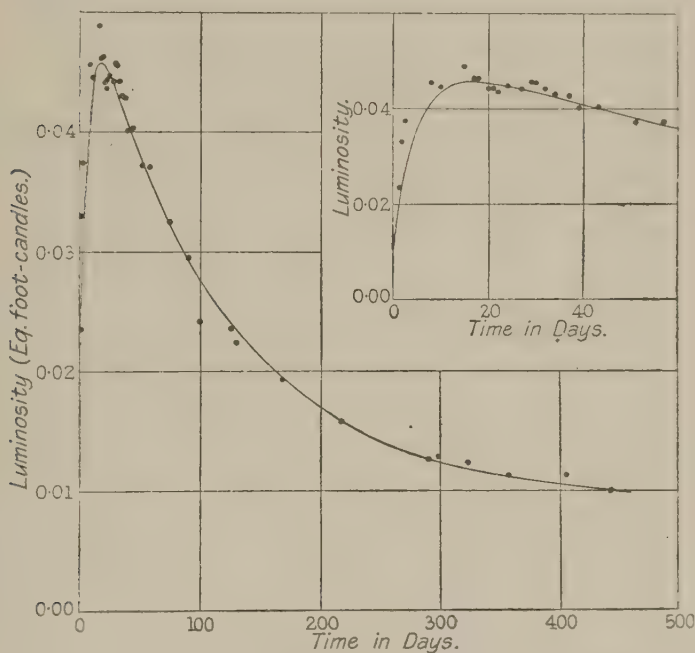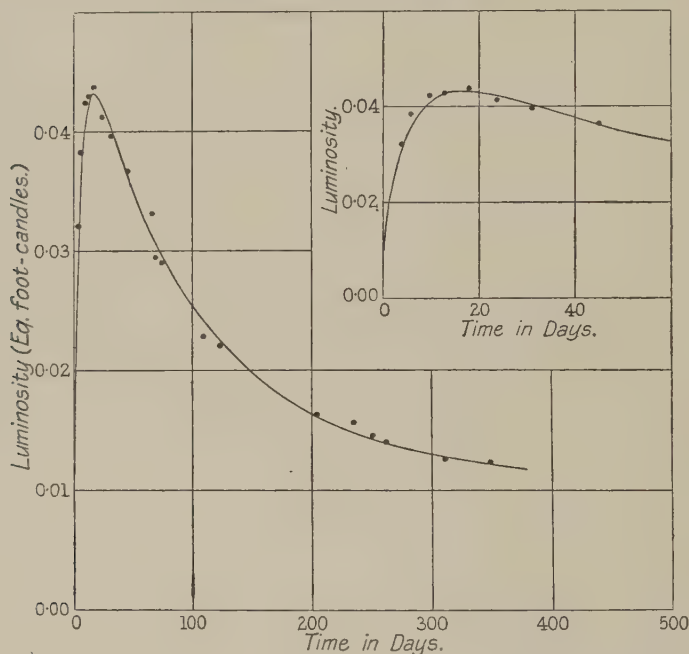

FIG. 8.—DIAGRAM SHOWING THE VARIATION THROUGHOUT LIFE OF THE LUMINOSITY OF SAMPLE 8 OF RADIUM LUMINOUS COMPOUND. INSET, THE EARLY PORTION OF THE LIFE ON AN ENLARGED SCALE.

about 15 months, and curves showing the rise and fall with time of the luminosity of two characteristic samples are shown in Figs. 7 and 8.

In samples 3 and 8 (Figs. 7 and 8) measurements were possible within a short time after mixing the radium salt with the zinc sulphide. The curves referred to above have an inset showing in greater detail the earlier history of these samples. It is seen that there is a growth of luminosity to a peak value which is attained generally about 15 days after mixing. The

characteristics of the early life of luminous compound are of special interest as bearing on the theory underlying the action of radium on zinc sulphide.

The values of luminosity observed during the first 40 days are taken from the curves and scheduled in the following table :—

TABLE IV.

Period after mixing. Days.	Luminosity (equivalent foot-candles).			
	Sample 3.		Sample 8.	
	Actual.	P.c. of maximum.	Actual.	P.c. of maximum.
4	0·032	71	0·030	70
6	0·038	84	0·035	81
8	0·041	91	0·039	91
10	0·043	95	0·041	95
15	0·045	100	0·043	100
20	0·045	100	0·043	100
25	0·044	98	0·042	98
30	0·043	96	0·040	93
35	0·042	93	0·039	91
40	0·041	91	0·037	86

It should be noted that these samples were from different makers, the method of mixing the radium salt with the zinc sulphide being stated to be different in the two cases. In the first sample the mixture was made with both salts in the dry state, while in the second the radium salt was added in the form of a solution to the zinc sulphide, the resulting mixture being then evaporated to dryness. With this latter method of mixing it is certain that nearly all the radium emanation is driven off, so that the subsequent increase of luminosity of the radium paint is due to the gradual growth of the radium emanation and its products radium A and radium C.

After attaining its maximum value at the end of about 10 to 20 days from mixing, the luminosity then begins to decrease, at first very gradually and then more rapidly until, after a period of about six to seven weeks from the date of mixing, the rate of decay appears to have settled down in accordance with an exponential law—i.e., the rate of decay (k) of luminosity (B) at any time is approximately proportional to the value of the luminosity at that time and $B/B_0 = e^{-kt}$. The logarithm of the luminosity therefore plotted against time should give a straight line. The logarithmic curve corresponding with the results of sample 3 (Fig. 7) is shown in Fig. 9. After

an initial period of 40 days it will be seen that the logarithmic relation given above is nearly fulfilled up to the end of about 150 to 200 days, but that after this period the luminosity does not diminish so rapidly as would be expected from the above law. It will be shown in a further Paper * that the effect may be explained on the assumption that superposed on the normal
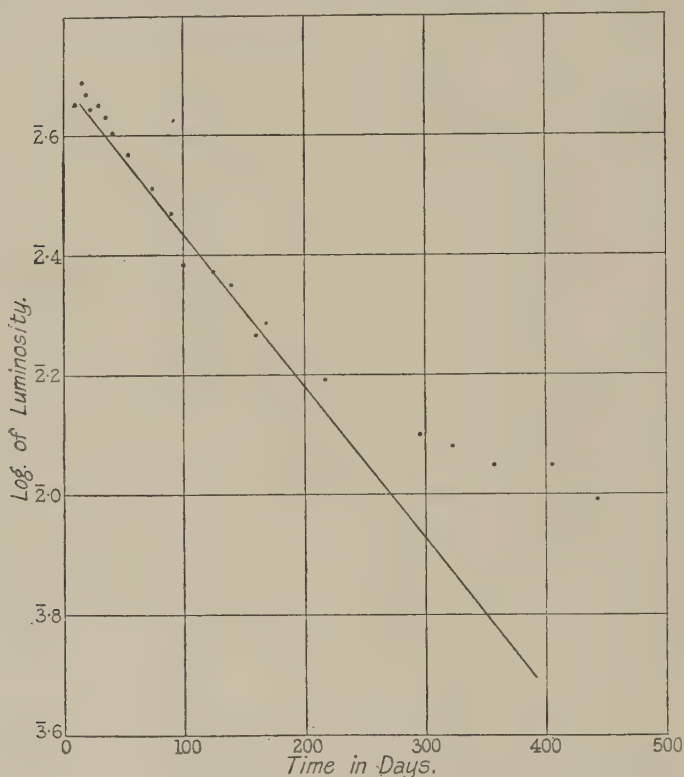
FIG. 9.—DIAGRAM SHOWING THE GRADUAL DEPARTURE OF THE OBSERVED VALUES OF LUMINOSITY OF SAMPLE 3 FROM THE LOGARITHMIC RELATIONSHIP.

decay proportional to the luminosity, there is a smaller effect operating in the reverse direction, which becomes relatively more and more important as the luminosity becomes smaller. The observations have not continued long enough to enable the relative importance of these effects to be determined with great precision, but the results point to the conclusion that after

* "The Theory of Decay in Radioactive Luminous Compounds," by J. W. T. Walsh.

about 500 days the luminosity will attain a value wh'ch is
sensibly constant. This period, however, is not the same for all
compounds, as in some of them the rate of decay appears to di-
minish more rapidly than in others. The effect on the rate of
decay of the amount of radium present is shown by the curves
in Figs. 10 and 11. Each of the two pairs of compounds 1
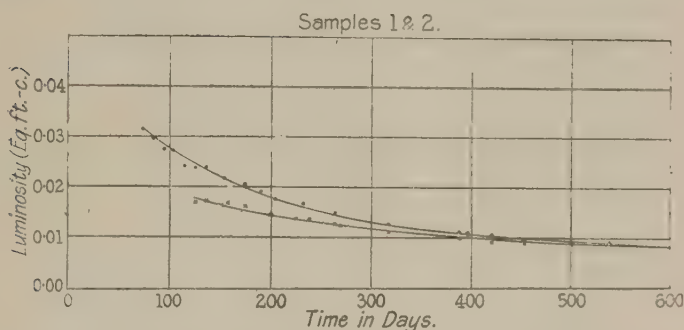
FIG. 10.—LUMINOSITY CURVES FOR SAMPLES 1 AND 2 OF DIFFERENT
RADIUM CONTENTS, SHOWING THE SLOWER RATE OF DECAY OF COMPOUND
CONTAINING HALF QUANTITY OF RADIUM.
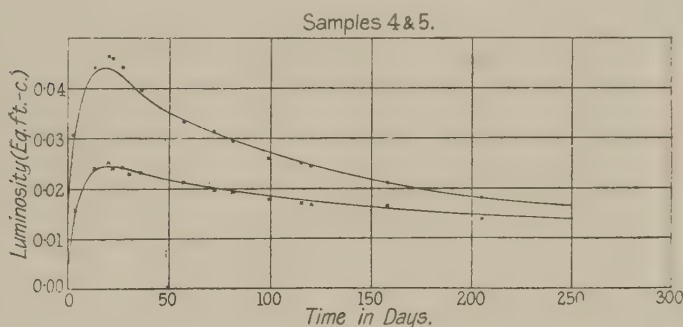
FIG. 11.—THE EARLY PORTIONS OF THE LUMINOSITY CURVES FOR SAMPLES
4 AND 5 OF COMPOUND CONTAINING RADIUM IN THE PROPORTION OF 2 : 1.

and 2 (Fig. 10) and 4 and 5 (Fig. 11) was made up at the same
time by the same manufacturer with an identical quality of
zinc sulphide. One sample of each pair was prepared with
exactly half the quantity of radium salt contained in the other.
In the case of the first pair, observations were not made until
about 100 days after mixing, while in the case of the second
pair the first five months' observations are all that have so far

been obtained.   It will be seen that the initial brightness is in
the ratio of the radium content, but the rate of decay of the
compound of smaller content is slower, so that the ratio of the
luminosities gradually approaches unity as time goes on.
This question is one of considerable commercial importance,
for it is clear that if the compound is required for use after the
lapse of several years, there is very little advantage, except
from the point of view of initial brightness, in increasing the
radium content beyond a certain point.   From present
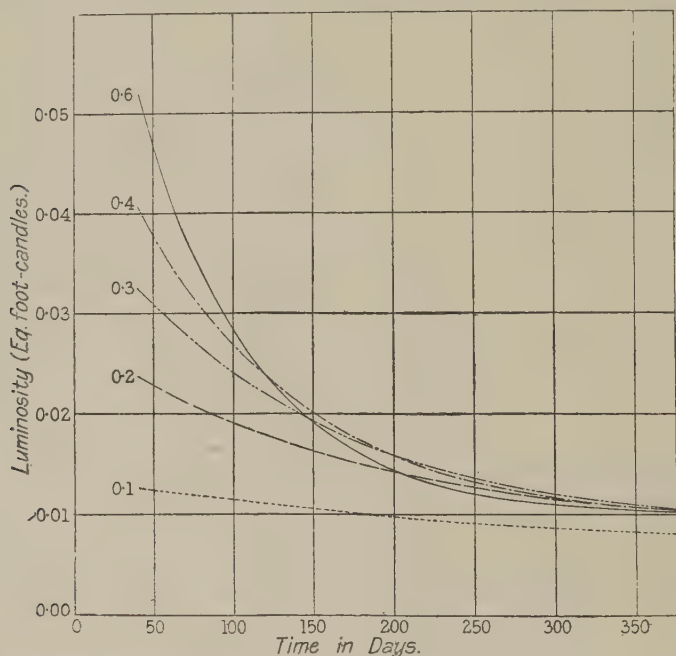
FIG. 12.—THEORETICAL LUMINOSITY CURVES FOR SAMPLES OF RADIUM
CONTENT (MG./GM.) SHOWN AGAINST EACH CURVE.   (CONSTANTS OF DECAY
ASSUMED OF FAIR AVERAGE VALUES.)

knowledge it would appear that for short period use (say three
to four months) the radium content may be usefully as high as
0·4 mg. per grm.   Fig. 12 shows a family of luminosity curves
based on the average data at present obtained for compounds
made up with different proportions of radium salt in the same
quality of zinc sulphide, this sulphide being assumed to have
characteristic constants of a fair average value.
    It is to be noted that in this diagram in which the effect of

the diminution of decay with time is allowed for, the curves for 0·4 and 0·2 mg. compound do not cross one another as would curves based on a purely logarithmic law with constant rate of decay throughout.

(b) *Painted Dials*.—Seven dials painted with different samples of 0·4 mg. luminous compound have been kept for the past 12 months, and their luminosities have been found to decay to approximately half value in that period. The actual results of the measurements are given in Table IV. The dials are all ordinary metal dials used for aeroplane instruments, painted with a matt black surface, and having the luminous compound laid on in fine l.nes about 0·2 mm. to 0·3 mm. high to mark the graduations and figures of the dial. These dials were tested in the apparatus described on p. 229, with the following results :—

TABLE V.

Dial No.	Initial luminosity (p.c. of standard).	Luminosity after 11 months (p.c. of standard).	Decay of luminosity (p.c. of initial luminosity).
1,406	94	60	63
995	82	46	51
1,470	94	58	62
1,783	63	34	54
1,794	24	(16)	(67)
1,496	91	40	44
1,497	52	25	48
		Average	55 p.c.

It is to be remarked that the rate of decay of luminosity of the dials is noticeably lower than that of the compound before application, and as has been said, the initial luminosity of the painted dial is also smaller, being of the order of one-third to one-fourth of that of the compound before application. The reason for this is not clear, but it seems reasonable to suppose that the effect of the varnish or other medium is to impede the activity of the α-ray bombardment from the radium and in this way to cause an effective lowering of the radium content of the compound with a subsequent reduction of both the initial luminosity and also the rate of decay. It is not suggested that the whole effect is due to such a cause, but it seems likely that some such effect as this is operative to account for at least a portion of the observed effect. It is significant, for instance, that the luminosity standards for the dry compound and for the painted dials are 0·03 and 0·0075 ft.-candles respectively

(*i.e.*, in the ratio of 4 : 1), while, if we assume curve 0·4 of Fig. 12 to represent the behaviour of the compound before application to dials, the relative behaviour *after* application would be very well represented, both as regards initial brightness and rate of decay, by a curve lying between those for 0·2 and 0·1 of the same diagram.

7. *Considerations Governing the Choice and Specification of Radium Luminous Compound.*

(*a*) *Radium Content.*—It will be clear from what has been stated above concerning the relation between the radium content of a compound and the rate of decay of its luminosity, that, in specifying a standard for any particular purpose, the probable life of the compound in actual use must be the principal deciding factor as to the particular radium content to be employed. The factor of cost, however, must not be ignored. Compound containing 0·4 mg. of radium per gramme of zinc sulphide costs about £6 to £7 per gramme. This is mainly due to the amount of radium present, and the cost may, therefore, be considered as roughly proportional to radium content.

Further, from what has been said above (p. 239) as to the relative rates of decay of radium compound in powder form and after application to dials, it will be seen that in considering the standards to be adopted in the two cases, the radium content of compound *after application* must be considered as reduced to about one-third, in order to ascertain its rate of decay from the data given in the diagram of Fig. 12, p. 238, which refer to the powder only ; that is to say, a *dial* painted with compound containing 0·3 mg. per grm. of radium will follow approximately the same decay curve as *powder* containing 0·1 mg. per grm.

For the compound alone (as distinct from painted dials) after the first six months, the luminosities of compounds of all contents between 0·2 mg. per gramme and 0·8 mg. per gramme are the same to within 20 per cent.

Considering, first, the upper limits desirable for radium content, it is seen that if the compound is likely to be required for more than six months, the use of a compound of radium content greater than 0·2 mg. per gramme can only be justified by a need for greater brilliancy during the early part of the life. Similarly, it may be said that for painted dials the maximum useful radium content for a life of over six months will not be more than 0·6 mg. per gramme.

In the case of instruments of which the life under normal working conditions is appreciably less than six months the use of a compound containing, say, 0·6 mg. per gramme or over might be of advantage, provided it could be ensured (a) that the dials were painted within three weeks of the preparation of the compound, and (b) that the instruments were not kept in store for more than a month after painting. Probably, in ordinary circumstances, these conditions would not be easy of realisation, and the use, therefore, of a compound of greater radium content than about 0·4 mg. per gramme may be ruled out. Even for 0·4 mg. compound it is important that the interval between the preparation of the compound and its application to the dial should be as short as possible, for it is clear that deterioration in luminosity takes place from twice to three times as rapidly in the powder as in the painted dial. Thus, a deterioration due to one month's delay in painting will be as much as that resulting from about three months' storage of the completed dial.

Referring next to the lower limits desirable for radium content, we see that for cases in which the compound is employed in powder form, the minimum useful radium content is only a little below 0·2 mg. per gramme, and except on the score of expense alone it is not desirable to employ a compound of less radium content than this, because the luminosity of the 0·1 mg. compound after an indefinitely long period, is still nearly 20 per cent. below that of the 0·2 compound. Hence, even in cases where very long life is desirable without the possibility of replacement, the use of the higher grade material is to be preferred unless the cost of the radium compound used has a preponderating influence on the price of the complete instrument. In such cases, rather less than half the cost would be saved and 20 per cent. of the final luminosity would be sacrificed by the adoption of 0·1 mg. per gramme for the compound in powder form.

Similarly for compound used in painting dials, 0·4 mg. per gramme is the best content to use if final luminosity is not to be sacrificed. Some 50 per cent. saving of cost should result from the use of compound for painting dials containing 0·2 mg. per gramme, but the final luminosity would be somewhat less. It should be remembered that there is nothing appreciable to be gained by the use of compounds for dials *below* this last named limit because it will be nearly as economical, and a better result will be achieved by employing a smaller quantity

of the better grades of compound instead of the large quantity of a compound of lower radium content.

(b) *Luminosity.*—The standard of luminosity for the luminous compound in powder form should be so fixed as to enable a good grade of compound to pass the test within a reasonable period after its preparation. The present standard for 0·4 mg. compound (*viz.*, $0·03_0$ f.c.) is easily attainable in a good grade three months after preparation. This time limit may be extended for grades of lower content to, say, four months for the 0·2 mg. compound and six months for 0·1 compound. The corresponding luminosity standards for the three grades 0·4, 0·2 and 0·1 are then 100, 60 and 35 per cent. of the present National Physical Laboratory standard. For painted dials the standards of luminosity should be 25 per cent. of the standards fixed for the powders from which they are to be painted. The above-named figures of 100, 60 and 35 per cent. will then express the standards of the painted dials in terms of the present National Physical Laboratory standard for dials—*i.e.*, 0·0075 equivalent ft.-candles.

The above conclusions have been based on the results obtained by observation of the samples of paint at the National Physical Laboratory from October, 1915, to January, 1917. None of these compounds have a radium content greater than 0·4 mg. per gramme, so that the behaviour of such grades of compound can only be predicted on the assumption that they will obey the same laws as those which have been found to hold for the samples of lower radium content. Further, a series of observations extending over several years would be necessary for the certain prediction of the behaviour of compound for long periods. It has been necessary, however, to make decisions from the results already obtained and the laws deduced from them, without the long delay necessary for experimental confirmation. The conclusions given above are those which experiment has so far shown to be most likely to be justified.

These conclusions are briefly summarised in the two following tables.

C ASE A.—*Compound Used in Powder Form.*

Period of enhanced luminosity.	Radium* content.	Luminosity standard.
About six months ...	0·3 mg./gm.	80 p.c. of N.P.L. standard for compound
About one year	0·2 mg./gm.	60 p.c. „ „

* Expressed in gamma ray activity of an equivalent weight of hydrated radium bromide.

CASE B.—*Compound Used for Painting Dials.*

Period of enhanced luminosity.	Radium content.	Luminosity standard.		
About six months ...	0·6 mg./gm.	140 p.c. of N.P.L. standard for compound		
About one year	0·4 mg./gm.	100 p.c.	,,	,,
Sensibly uniform luminosity with a final value about 30 p.c. below that of the other grades	0·2 mg./gm.	60 p.c.	,,	,,

The authors desire to express their acknowledgments to Capt. Creagh-Osborne, Admiralty Superintendent of Compasses, and his assistant, Lieut. Bray. They also wish to express their warm appreciation of the co-operation of manufacturers and others in the work which has been done, and particularly that of Mr. F. H. Glew and Mr. H. A. Hughes.

APPENDIX A.

ABSORPTION OF γ-RAYS BY THE ZINC SULPHIDE IN RADIUM LUMINOUS COMPOUND.

(*a*) *Approximate Graphical Method.*—The first step is to find the difference between the emergent intensities for axial con-
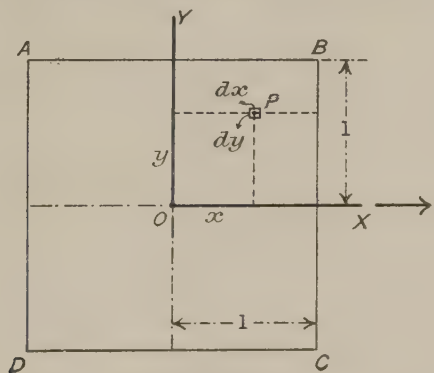

FIG. 13.

centration and for uniform distribution of radium in the case of a vessel of square section.

Let $A\,B\,C\,D$ be a section of the vessel of side $2l$, 0 being the

axis. Then, for axial concentration the intensity in the direction of the OX axis is

$$Ie^{-\lambda l} = I\left\{1 - \lambda l + \frac{\lambda^2 l^2}{2!} - \quad . \quad . \quad . \quad . \quad . \quad .\right\}$$

For uniform concentration, however, consider the elementary particle P. The intensity due to this particle P of area $dxdy$ is

$$\frac{dxdy}{4l^2} . I . e^{-\lambda(l-x)}.$$

Hence, the total intensity for the whole section is—

$$\left(\frac{I}{4l^2}\right)\int_{-l}^{+l}\int_{-l}^{+l} e^{-\lambda(l-x)}dxdy$$

$$= \frac{I}{2\lambda l}(1 - e^{-2\lambda l}),$$

$$= I\{1 - \lambda l + \tfrac{2}{3}\lambda^2 l^2 - \quad . \quad . \quad . \quad . \quad .\}$$

It will be seen that the two expressions obtained above are equal to $0\cdot1$ per cent. if $\lambda l < 0\cdot1$, while if $\lambda l < 0\cdot01$ the expressions are identical to 10 parts in a million.

Fig. 14.

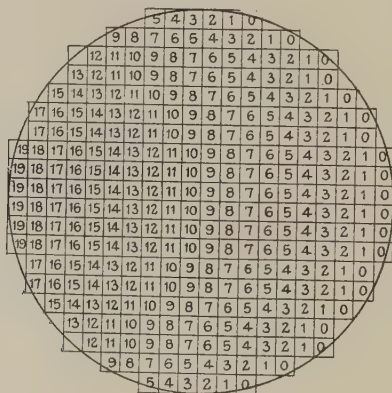Now, λ has been experimentally determined by placing a small tube of radium axially in a cylindrical vessel of zinc sulphide, of which the diameter was 24 mm. The intensity thus found was $0\cdot895\,I$, so that

$$e^{-12\lambda} = 0\cdot895,$$

giving $\qquad\qquad \lambda = 0\cdot00925 \text{ mm}^{-1}.$

Now, consider the case of a cylinder of radius a, and let this

be divided into a number of small vessels of square section of side $2b = a/10$, as shown in Fig. 14.

It is clear that in this manner uniform distribution over the cylinder may be considered as equivalent to that due to concentration at the axes of 156 square vessels of side $a/10$. Thus the intensity in the direction of the arrow due to each vessel marked " O " will be $(I/312)e^{-\lambda b}$, that due to each vessel marked " 1 " will be $(I/312)e^{-3\lambda b}$, and so on. In this way it is found that for a vessel of 24 mm. diameter, the intensity s $(0.910)\,I$, instead of the $(0.895)\,I$, due to axial concentration.
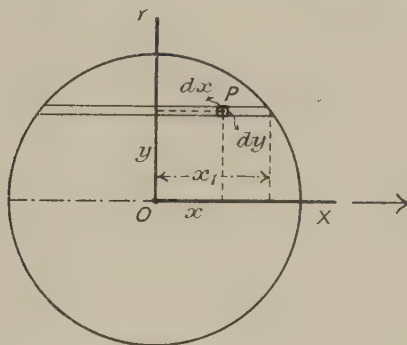
FIG. 15.

(b) *Rigid Mathematical Determination.*—Let O be the axis of a cylindrical vessel of radius a, and let P be an elementary particle of area $dx\,dy$. Then the intensity along the OX axis due to the particle at P will be

$$\frac{I}{\pi a^2}dx\;dy\;e^{-\lambda[\sqrt{a^2-y^2}-x]}.$$

Hence, the total intensity is—

$$I' = \frac{I}{\pi a^2}\int_{y=-a}^{y=+a}\int_{x=-\sqrt{a^2-y^2}}^{x=+\sqrt{a^2-y^2}}e^{-\lambda[\sqrt{a^2-y^2}-x]}dx\;dy,$$

$$= \frac{2I}{\lambda\pi a^2}\int_0^a (1-e^{-2\lambda\sqrt{a^2-y^2}})dy.$$

Expanding the exponential and integrating each term this gives—

$$I - \frac{\lambda}{a}\cdot\frac{\pi a^2 I'}{2I} = 1 - 2\lambda a\cdot\frac{\pi}{4} + \frac{4\lambda^2 a^2}{2!}\cdot\frac{2}{3} - \quad.\quad.\quad.\quad.\quad.\quad.$$

$$+ (-1)^r\frac{(\lambda a)^r}{r!}\cdot\frac{\Gamma\{(r+2)/2\}}{2\Gamma\{(r+3)/2\}} + \quad.\quad.\quad.\quad.$$

This gives—

$$I'/I = 1 - \frac{8}{3\pi}(\lambda a) + \tfrac{1}{2}(\lambda^2 a^2) - \frac{32}{45\pi}(\lambda^3 a^3) + \quad . \quad . \quad . \quad .$$

It will be seen that this series is rapidly converging for all values of λa up to unity, and for the sake of convenience the values of the first 12 coefficients are tabulated below. These will enable the value of I'/I to be calculated correct to $0\cdot01$ per cent. for all values of λa up to unity.

Table.

The first 12 coefficients in the expression for the γ-ray activity of a sample of radium compound contained in a cylindrical vessel, viz. :—

$$I'/I = z_0 + z_1(\lambda a) + z_2(\lambda^2 a^2) + \quad . \quad . \quad . \quad . \quad .$$

Coefficient.	Value.	Coefficient.	Value.
Z_0	$+1\cdot000000$	Z_7	$-0\cdot001642$
Z_1	$-0\cdot848826$	Z_8	$+0\cdot000347$
Z_2	$+0\cdot500000$	Z_9	$-0\cdot000067$
Z_3	$-0\cdot226354$	Z_{10}	$+0\cdot000012$
Z_4	$+0\cdot083333$	Z_{11}	$-0\cdot000002$
Z_5	$-0\cdot025869$	Z_{12}	$+0\cdot000000$
Z_6	$+0\cdot006944$		

Assuming the value for λ given above $\lambda a = 0\cdot111$ for $a = 12$, so that $I'/I = 0\cdot9116$, with which the value $0\cdot910$ found by the graphical method is in good agreement.

ABSTRACT.

The Paper contains the results of measurements made on various samples of radium luminous compound during the last two years. Determinations of the brightness of the compound in powder form and when made up into paint, and also after the application of the paint to instrument dials, were carried out ; and curves are given showing the rates of decay of luminosity. The radium contents of the compounds were determined by comparison of their γ-ray activities with that of a preparation of pure radium bromide, which is periodically compared with the British radium standard. The various precautions which have to be observed, and the corrections which have to be applied in making the various determinations are explained, and the considerations which should govern the proportion of radium employed for practical purposes are discussed.

DISCUSSION.

Mr. F. H. GLEW : The difference between some of Marsden's results and those of Dewar and Crookes had been attributed to some difference in the zinc sulphide employed : he could state that the sulphide supplied to the National Physical Laboratory was of different origin from either of these. Before the war we were almost entirely dependent on foreign sources for the supply of these materials, but we are now quite self-supporting. In

Marsden's work the inside of the tube was completely coated with sulphide, so that the effects were observed by light transmitted through the material. It was better to paint only one half of the tube so that the inner surface, which was exposed to the bombardment, could be directly observed. In the photometer measurements, was there any advantage in placing the coloured filter between the light and the reflecting surface ? He preferred to have it in the eye-piece. Earlier records of the coloration due to absorption would be of value. In testing sulphide for γ-ray activity he placed it in contact with the ionisation chamber. Less material was then required for the test, which was an advantage. The authors state only the diameters of the tubes employed ; but he had found that there were differences in the results obtained with tubes horizontal or vertical. The length of the tube, therefore, appeared to be an important factor. Further, the results given by a small quantity of sulphide in a large tube would be too high ; for the space above the sulphide is filled with the emanation gas, and the γ radiation from it is not subject to the absorption of the sulphide, as is the radiation from the latter itself. The discrepancy between the observed and calculated values of the absorption was probably important. Was this due to some selective action, the sulphide transforming the γ radiation into radiation of some other type ? It was singular that specimens prepared by the dry and wet methods should be so similar in their properties after only 4 days ; for in the dry method the emanation and all its products are retained.

Dr. MAKOWER stated that by using an electroscope of small capacity good γ-ray measurements could be obtained with much less than 1 milligram of radium.

Dr. H. S. ALLEN asked if the zinc sulphide employed was wholly crystalline or partially amorphous. An investigation of its physical condition would be useful.

Prof. CLINTON said that the difficulties of these measurements were very great and the authors were to be congratulated on having overcome them so effectively.

Mr. J. S. DOW said the Paper contained a great deal of useful information. It was common knowledge that a great deal of work on these luminous paints was being done, and they were fortunate in having these results made public. No doubt after the war even fuller information would be forthcoming. His own experience was generally in accord with a number of important results announced by the authors. For example, he found that freshly-prepared compound of 0·4 composition had a brightness not far removed from that mentioned. He rather doubted whether the absolute brightness could be given with exactitude, though an empirical result could no doubt be reproduced quite closely in the laboratory. He had noticed, for example, that if two patches of zinc sulphide and calcium sulphide were excited by visible light, their brightness might be equal when viewed at a distance of a few inches ; but the zinc sulphide (giving a green light) appeared much the brighter as the distance of the observer is increased. There was also the question whether the colour-match obtained by using a blue-green glass with an incandescent lamp was sufficiently close to eliminate the purkinje effect at low illumination. With low luminosity materials he had found that a slight obliquity of gaze affected the comparison. He agreed that the colour required was a green with a distinctly blue component. It was curious, however, that while with bright specimens any colour departure was evident, almost any green glass appeared to give a colour match at weak luminosities. He had also found that the colour of freshly-prepared and very bright samples was distinctly more blue in tint than that of old and weak ones. The authors' suggestion that the most satisfactory compound, from the standpoint of luminosity in the later period of life and minimum cost, was the 0·2 % material, was a most important result. He thought this would probably be borne out by later experience. In view of the expense, it was

obviously necessary to make the radium content as small as possible. Another point on which little information was forthcoming was the order of luminosity necessary in practice for different purposes. Such consideration would probably also favour the use of 0·2 material in most instances. In his experience, in complete darkness, much brighter material might actually cause irradiation. He would like to ask whether the authors had found that in the weaker compounds the initial rise in brightness was disproportionately great and more prolonged than would be expected from the brighter specimens. This was a feature of some tests of which he had received particulars from the United States, and was in some measure supported by his own experiments. Finally, it might be of interest to refer to a series of observations he had recently made of the brightness of a tube coated internally with zinc sulphide and filled with radium emanation, which Mr. Glew had kindly placed at his disposal for test. The initial brightness of this tube
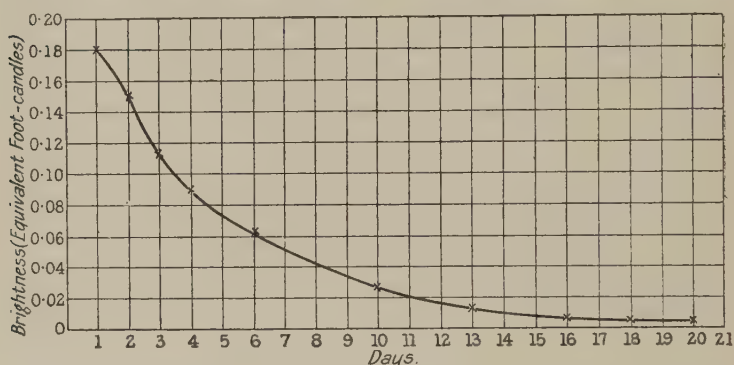
DECAY OF LUMINOSITY OF TUBE FILLED WITH RADIUM EMANATION AND COATED INTERNALLY WITH ZINC SULPHIDE IN THE COURSE OF THREE WEEKS AFTER PREPARATION.

was relatively high—of the order of 0·2 foot-candles ; but in three weeks it fell to 1/60th of its initial value. Naturally such a tube would decay much more rapidly than a compound containing powder. Another feature was the very clear exhibition of the formation of local patches of brightness due to condensation of the emanation, to which Mr. Glew had referred. Patches, constantly changing their position, as bright as four times the value of the main luminosity would make their appearance, and then fade away and reappear in another part of the tube. This effect was marked in the early stages of the life, but much less evident at a later stage, when the general brightness had diminished. It would be interesting to know if such effects were met with in varnish or powder materials. In his personal experience he had met only one case, an exceptionally bright tube which was initially much brighter at one end than the other. A few days after the first test the inequality disappeared and had not been noted since.

Dr. LEVY said he had found the luminosity of different samples of zinc sulphide to vary perceptibly in tint when stimulated by daylight, but had not noticed this with α-ray stimulation. In his experience, if a sample of sulphide gave good results when stimulated by daylight, it would also be good when stimulated by radium. The reverse did not hold, however, as a specimen might be poor with daylight, but good with radium excitation.

Mr. T. SMITH thought the coloured filter used in the photometric measurements should have been placed as near the lamp as possible,

since such a filter acts, to a certain extent, as a self-luminous source. If the tube containing the compound was not much wider than the aperture in the screen, a considerable loss of light would occur, near the edges of the aperture, due to oblique internal reflection at the walls of the tube. For accurate work it was important to have a uniformly illuminated field, and he suggested the desirability of employing tubes with a flat side. Where the tubes were too small to fill the aperture s tisfactorily, an optical system could be used to give an enlarged image of the tube and so avoid refractions at large angles. The absorption of the system could be found experimentally with a larger tube.

Dr. A. RUSSELL laid stress on the practical importance of the Paper. He had at times been asked by manufacturers to measure the visibility of watch faces in the dark. The problem was so difficult that hitherto he had done nothing. The present Paper suggested the required practical tests, and a first draft of a specification for securing a standard material could now be written. Progress would be delayed if they waited till all the associated physical and physiological problems were cleared up.

Dr. S. RUSS (in a communication which was read by the Secretary) said the decay curves differed appreciably from those of Marsden for Willemite under intense a-ray bombardment. Did the theory held by the authors as to the mechanism of the action differ from that advanced by Prof. Rutherford to explain Marsden's results ? Had the authors any measurements of the rate of decay of efficiency of fluorescent screens under the action of X-rays ?

Mr. WALSH, in reply, said that if the coloured filter were employed in front of the eye, as suggested by Mr. Glew, its coefficient of transmission for the light emitted by the compound would require to be determined and introduced in the results as well as the coefficient for white light This would add to the uncertainties of the determinations. To detect change of colour with age would have been difficult in their experiments because of the continuous decrease in luminosity. They had detected differences in colour in samples sent by different makers ; but were unaware whether these were from the same source, or made in the same way. The authors had no data on the relation between daylight and radium excitation, and Dr. Levy's remarks were of great interest. The order of accuracy (about one per cent.) was not found to be appreciably affected by altering the position of the coloured screen. The diameter of the smallest tubes usually employed was not less than about twice the apparent width of the aperture. It was not convenient to use a smaller aperture, and they had to do their best with the tubes in which the material was sent. They had not found any samples of radium compound which gave the slow initial rise of luminosity mentioned by Mr. Dow. It is possible, however, that if some radio-active agent other than radium were employed, such, for example, as mesothorium a slow rise of luminosity might very likely result. They would be interested to have further details of the samples referred to by Mr. Dow in this connection.

Mr. HIGGINS, replying to Dr. Makower, said that they had to employ the radium standards which were available, and these were not suitable for use with a small capacity electroscope. In reply to Mr. Glew, the tube should not be close to the electroscope. If it was 30 to 40 cm. off, the orientation of the tube was of little importance, and small corrections could be applied. The errors introduced by having it too close were similar in character to those produced if an extended source of light is used too near a photometer head. They suspected selective action to be at the root of the discrepancy in the absorptions, but could not definitely say. To investigate the physical properties of the sulphide would be a very long task, and they had not attempted it in the present research.

XIX. *The Resistance to the Motion of a Lamina, Cylinder or Sphere in a Rarefied Gas.* By F. J. W. WHIPPLE.

In Dr. P. E. Shaw's Paper* on the measurement of the pressure in a high vacuum by observation of the logarithmic decrement of the motion of a suspended system, use is made of a formula, due to Prof. Poynting, for the resistance experienced by a plane lamina moving through a rarefied gas. Dr. Shaw mentions that Prof. Poynting had proposed the formula as indicating the order of magnitude of the resistance, and that a numerical factor was possibly to be inserted. A closer examination of the problem and of the validity of its adaptation to the case of a moving cylinder appears to be desirable.

In the rarefied gas with which Dr. Shaw is concerned the free paths of the particles are long compared with the diameter of the containing vessel, so that the velocities are distributed at random—*i.e.*, the velocities of the particles which a lamina meets in any small interval of time are not affected by concussion with particles which have been reflected from it already. In the following investigation it is assumed that the velocities of the particles are distributed according to the Maxwellian law, though it may be questioned whether that law holds good in a gas when collisions with the boundary are more frequent than collisions amongst the particles.

The specification of the conditions of the impact between a particle and the lamina presents special difficulties. Dr. Shaw's argument assumes that the gaseous particle behaves like a sphere rebounding from a perfectly elastic plate. Actually we have to deal with the movement of a molecule which comes into contact with one or more of the molecules of the plate, which molecules are themselves in movement, and there is an interchange of energy.

In this connection reference may be made to a series of papers by M. Knudsen, of Copenhagen. The problems discussed by Knudsen are connected with diffusion and viscosity, and he finds that the assumption † that the directions of motion of the gas particles rebounding from a solid boundary are distributed

* " Proc." Phys. Soc., 1916, p. 171.
† " Ann. d. Physik," XXVIII., 1, pp. 75-130. I am indebted to Dr. Chapman for this reference.

at random and independent of the directions in which they approach it is consistent with the experimental evidence. The velocities of the rebounding particles depend only on the temperature of the boundary.*

The corresponding assumption in the case of a moving solid is that the velocities of the rebounding particles *relative to the solid* are determined by the temperature of the solid, and are not affected by its motion. The molecules of the gas may be thought of as absorbed and emitted again, and the pressure as made up to two parts, corresponding with the impulses which reduce the molecules to relative rest and to the impulses which drive them outwards from the surface.

Resistance to the Motion of a Lamina.

If ρ_0 be the density of the gas as a whole and V the standard speed of the molecules, the density of those having speeds between u and $u+du$, and moving in directions with polar co-ordinates from θ to $\theta+d\theta$ and azimuths φ to $\varphi+d\varphi$ is

$$d\rho = \rho_0 \frac{\sin\theta d\theta d\varphi}{4\pi} \sqrt{\frac{54}{\pi}} e^{-3u^2/2V^2} \frac{u^2 du}{V^3}.$$

We consider in the first instance a lamina moving at right angles to its plane with speed v and measure θ from the direction of motion of the lamina.

The relative velocity with which a particle approaches the lamina has a normal component $u\cos\theta + v$. The total mass of the particles with velocities within the prescribed limits and striking unit area of the lamina in unit time is $(u\cos\theta + v)d\rho$. The corresponding pressure is $dp = (u\cos\theta + v)^2 d\rho$.

The particles hit the positive side of the lamina—*i.e.*, the side to which it is moving—provided that $u\cos\theta + v$ is positive. The limiting value of θ is α given by $\cos\alpha = -v/u$.

Evidently α is a little greater than a right angle. The total

* It should be mentioned, however, that in a later Paper ("A.d.P.," XXXIV., 4) Knudsen finds it necessary to introduce a "coefficient of accommodation," which permits the hypothesis that the energy of the rebounding particles depends in part on the energy with which they approach the boundary. The introduction of such a coefficient would increase slightly the reactions found in the present Paper. The hypothesis of Smolu-chowski ("Contributions to the Theory of Transpiration, &c., in Rarefied Gases," "Bull." International de l'Acad. des Sciences de Cracovie, 1910, A. No. 7, p. 295) is similar to Knudsen's. Cf. Soddy & Berry, Roy. Soc. "Proc." 1910, A., Vol. LXXXIV., p. 584.

pressure due to the impacts on the positive side of the lamina is
p_1 given by

$$p_1 = \frac{\rho_0}{4\pi} \int_0^\infty \int_0^{2\pi} \int_0^a (u\cos\theta + v)^2 \sin\theta\, d\theta\, d\varphi \sqrt{\frac{54}{\pi}}\, e^{-3u^2/2V^2} \frac{u\, du}{V^3}$$

$$= \sqrt{\frac{27}{2\pi}} \rho_0 \int_0^\infty \frac{(u+v)^3}{3} e^{-3u^2/2V^2} \frac{u\, du}{V^3}.$$

The corresponding pressure p_2 on the back of the lamina
can be written down in the same way. No particle whose
velocity is less than v will meet the lamina on this side so that

$$p_2 = \sqrt{\frac{27}{2\pi}} \rho_0 \int_v^\infty \frac{(u-v)^3}{3} e^{-3u^2/2V^2} \frac{u\, du}{V^3}$$

$$= \sqrt{\frac{27}{2\pi}} \rho_0 \int_0^\infty \frac{(u-v)^3}{3} e^{-3u^2/2V^2} \frac{u\, du}{V^3} + \varepsilon_2,$$

where

$$\varepsilon_2 = \sqrt{\frac{27}{2\pi}} \rho_0 \int_0^v \frac{(v-u)^3}{3} e^{-3u^2/2V^2} u \frac{du}{V^2},$$

$$< \sqrt{\frac{27}{2\pi}} \rho_0 \int_0^v \frac{v^4}{3} \frac{du}{V^3} \quad \text{or} \quad \sqrt{\frac{27}{2\pi}} \rho_0 \frac{v^5}{15 V^3}.$$

The resultant difference of pressure on the two faces of the
lamina is—

$$p_1 - p_2 = 2\sqrt{\frac{27}{2\pi}} \rho_0 \int_0^\infty \left(u^2 v + \frac{v^3}{3}\right) e^{-3u^2/2V^2} u \frac{du}{V^3} - \varepsilon_2,$$

$$= 2\sqrt{\frac{2}{3\pi}} \rho_0 \left(vV + \frac{v^3}{2V}\right) - \varepsilon_2.$$

As v is small compared with V, the terms in v^3 and v^5 may be
ignored, and the effective pressure corresponding with the
impulses which reduce the particles to relative rest is therefore

$$2\sqrt{\frac{2}{3\pi}} \rho_0 v V.$$

To find the pressure corresponding with the impulses which
set the particles in motion again we require the difference
between the masses emitted from unit areas on the two sides of
the lamina in unit time and the mean relative velocity normal
to the surface.

The statement that the difference in question is equal to
$v\rho_0$, the mass of gas which would be swept up by the lamina
in unit time if the particles were not in motion is plausible, and
is borne out by integration of expressions similar to those in the
preceding paragraphs.

To determine the mean relative velocity normal to the surface we assume that the law of distribution of relative velocity is the same as in the case of the fixed boundary of an enclosure containing gas at uniform temperature. In that case the distribution of velocities of the particles leaving the boundaries is such as to maintain the Maxwellian distribution throughout the gas.

In such a case the mass striking unit area of the wall in unit time is—

$$\iiint u \cos \theta . d\rho = \frac{V \rho_0}{\sqrt{(6\pi)}},$$

and the impulse given to this mass as it leaves the surface is—

$$\iiint (u \cos \theta)^2 d\rho = \frac{V^2 \rho_0}{6},$$

so that the mean normal velocity with which a particle leaves the surface is $\sqrt{\frac{\pi}{6}} V$.

Accordingly in the problem of the moving lamina the resultant pressure due to the impulses driving the particles outwards from the surfaces is

$$\sqrt{\frac{\pi}{6}} V v \rho_0.$$

It has been shown already that the pressure due to the impulses which reduced the particles to relative rest was

$$2 \sqrt{\frac{2}{3\pi}} V v \rho_0.$$

The effective force per unit area is therefore

$$\left(\sqrt{\frac{\pi}{6}} + 2\sqrt{\frac{2}{3\pi}} \right) V v \rho_0,$$

or $\qquad \frac{4+\pi}{\sqrt{(6\pi)}} V v \rho_0.$

The pressure which the gas exerts on the boundary is p_0, given by

$$p_0 = \tfrac{1}{3} \rho_0 V^2,$$

so that the effective force on the lamina is

$$(4+\pi) \sqrt{\frac{3}{2\pi}} \frac{v}{V} p_0.$$

It will be seen that if the lamina had been assumed to be

perfectly elastic * the factor $4+\pi$ would have been replaced by 8.

Turning now to the problem of the motion of a lamina in any direction, we may resolve its velocity into components v perpendicular to its surface and w parallel thereto.

In specifying the direction from which a particle is approaching the lamina, we take θ as measured from the direction of v and the azimuth φ as measured from the plane containing v and w. If u be the speed of a molecule, the component of the relative velocity in the direction opposite to w is

$$u \sin \theta \cos \varphi + w.$$

R_1, the resultant stress parallel to the plane caused by impacts on its positive side, is given by

$$R_1 = \frac{\rho_0}{4\pi} \int_0^\infty \int_0^{2\pi} \int_0^a (u \cos \theta + v)(u \sin \theta \cos \varphi + w) \sin \theta \, d\theta \, d\varphi$$
$$\sqrt{\frac{54}{\pi}} e^{-3u^2/2\,\Gamma^2} \frac{u^2 du}{V^3}.$$

On integration with respect to φ the term independent of w vanishes, and—

$$R_1 = \frac{\rho_0 w}{4} \int_0^\infty \frac{(u+v)^2 u}{V^3} \sqrt{\frac{54}{\pi}} e^{-3u^2/2V^2} \, du.$$

Dealing with the corresponding stress on the negative side in the same way and disregarding the terms of lower order as before we find that the effective traction is

$$\frac{\rho_0 w}{2} \sqrt{\frac{54}{\pi}} \int_0^\infty e^{-3u^2/2\,V^2} \frac{u^3 du}{V^3},$$

$i.e.,$ $\qquad \sqrt{\frac{2}{3\pi}} \rho_0 w V,$ or $2\sqrt{\frac{3}{2\pi}} \frac{w}{V} p_0.$

The velocities of the particles leaving the surface are distributed symmetrically round the normal, so that there is no corresponding tangential force. Accordingly, it follows that if a lamina is moving with such a velocity that the components at right angles to its surface and parallel to the surface are v, w respectively, the corresponding components of the resistance to its motion are

$$(4+\pi) \sqrt{\frac{3}{2\pi}} \frac{v}{V} p_0, \quad 2\sqrt{\frac{3}{2\pi}} \frac{w}{V} p_0 \text{ respectively.}$$

* As in the discussion on Dr. Shaw's Paper, *l.c.* p. 175.

Resistance to the Motion of a Cylinder.

We proceed to the case of a cylinder moving in a direction at right angles to its length. The cylinder is regarded as so long that the tractions due to the molecules of the gas on its ends may be ignored in comparison with the effects on the cylindrical surface.

The pressures and tractions on portions of the cylinder with parallel tangent planes may be combined as in the case of the lamina. The velocity of the cylinder can be resolved into components perpendicular to and parallel to any tangent-plane and the corresponding pressure and traction can then be resolved in the direction of motion. If D be the diameter of the cylinder and L its length, the total resistance it encounters when moving with speed v is seen to be

$$\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0 LD \int_0^{\pi} [(4+\pi)\cos^2\psi + 2\sin^2\psi]\frac{d\psi}{2},$$

or,

$$\frac{6+\pi}{4}\sqrt{\frac{3\pi}{2}}\cdot\frac{v}{V}p_0 LD.$$

In Dr. Shaw's Paper it was assumed that the resistance to the motion of a cylinder would be the same as the resistance to the motion of a lamina occupying the central plane of the cylinder. On the present hypothesis this is very nearly the case. The cylinder encounters more particles but, on the other hand, the impacts are not direct. The ratio of the resistance experienced by the cylinder and by the lamina of area LD is—

$$\frac{\pi}{4}\frac{6+\pi)}{(4+\pi)} \quad \text{or} \quad 1\cdot005.$$

Resistance to the Motion of a Sphere.

The case of the sphere can be treated in the same way as that of the cylinder. If a be the radius of the sphere and v is velocity, the resistance is—

$$\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0 a^2 . 2\pi \int_0^{\pi/2} [(4+\pi)\cos^2\psi + 2\sin^2\psi]\sin\psi d\psi,$$

or

$$(8+\pi)\sqrt{\frac{2\pi}{3}}\frac{v}{V}p_0 a^2.$$

The ratio to the resistance experienced by a plane lamina of area πa^2 is—

$$\frac{2}{3}\frac{(8+\pi)}{(4+\pi)} \quad \text{or} \quad 1\cdot04.$$

Damping of Motion in a Rarefied Gas.

When the formula obtained above for the resistance to the motion of a lamina is substituted for that given by Dr. Shaw, the coefficient K in the equation of motion is given by

$$K = \frac{(4+\pi)}{\sqrt{6\pi}} \frac{\rho V S a}{I},$$

and the density of the gas can be calculated from the formula

$$\rho = \frac{4\sqrt{(6\pi)}}{4+\pi} \frac{I\lambda}{V S a T},$$

or

$$\rho = 2 \cdot 43 \frac{I\lambda}{V S a T},$$

where S is the total effective area of the moving bodies, a the distance of their centres from the axis of rotation, I their moment of inertia, T the period of oscillation and λ the logarithmic decrement.

The coefficient $2 \cdot 43$ replaces the 3 of Dr. Shaw's Paper, so that the pressures calculated there should be reduced by about 20 per cent.

Added May 12*th*, 1917.

Motion in Connected Vessels.

The standard velocity of the gas-molecules can be eliminated from the resistance-formulæ by use of the equation $V^2 = 3R\theta_0$, in which R is the gas-constant and θ_0 the absolute temperature.

For example, the formula for the effective pressure on the lamina becomes

$$\frac{(4+\pi)v p_0}{\sqrt{(2\pi R\theta_0)}}.$$

The occurrence of the factors p_0 in the numerator and $\theta_0^{\frac{1}{2}}$ in the denominator is of interest, as it is known * that when two vessels at different temperatures and containing a rarefied gas are connected by a tube of narrow bore the pressures in the two vessels are not equal but related by the relation

$$\frac{p_0}{\theta_0^{\frac{1}{2}}} = \frac{p_1}{\theta_1^{\frac{1}{2}}},$$

p_0, p_1 being the pressures, θ_0, θ_1 the absolute temperatures.

* Soddy and Berry, *l.c.*, p. 583, quote Osborne Reynolds, "Phil. Trans." 1879, p. 727, and Knudsen, "Annalen der Physik," 1910, Vol. XXXI., p. 205.

Our formula shows that if equal laminæ are moving with equal speeds in two such vessels, the resistances to their motions will be equal. The oscillations of two suspended systems, such as Dr. Shaw's, made with the same dimensions would in like circumstances be damped at the same rate.

ABSTRACT.

The investigation is carried out on the assumptions that the free-paths of the particles of the gas are long compared with the dimensions of the moving body, and that the motion relative to the body of the particles which rebound from it depends only on its temperature. It is shown that if v, w be the components of velocity perpendicular to the surface of a lamina and parallel thereto the corresponding components of the resistance are

$$(4+\pi)\sqrt{\frac{3}{2\pi}}\frac{v}{V}p \quad \text{and} \quad 2\sqrt{\frac{3}{2\pi}}\frac{w}{V}p,$$

where V is the standard (root-mean-square) speed of the gas-particles and p is the gas-pressure.

The resistance to the motion of a cylinder or a sphere is found to differ very slightly from the resistance to a lamina occupying the central section. The formulæ are applicable to the problem of the damping of the oscillations of a system suspended in a rarefied gas.

DISCUSSION.

Mr. G. D. WEST called attention to the fact that the author's value for the ratio of the normal to the tangential resistance of a lamina, was in good agreement with that previously obtained by Knudsen. This experimenter had also deduced a formula for the resistance of a sphere moving in a rarefied gas, and it would be interesting to see how this result compared with the author's. In connection with the damping of vibratory motion, a large amount of careful work had been done by Hogg.

Dr. D. OWEN said that much interest now attached to the study of electrical phenomena occurring in the highest vacua. The present Paper was valuable as assigning a more precise figure for the constant to be used in reducing observations for pressure measurement by the method of damping. It should be noted, however, that the expressions in the Paper were arrived at on a particular assumption as to the nature of the recoil of the gas molecules, the experimental basis of which could not as yet be regarded as assured.

The damping method—which was used as far back as 1880 by Crookes—hardly seemed likely to be adaptable at pressures below $0.01\,\mu$ of mercury. It might, on the other hand, be useful in work at pressures well above $1\,\mu$, and it would be of practical value to know to what extent the formulæ given in the Paper would lead to error at higher and higher ratios of mean free path to diameter of vessel. For the lowest vacua Dr. Langmuir ("Phys. Rev.," 1913) has worked out a very promising method. A horizontal disc is driven by means of a rotating magnetic field at any speed up to 10,000 revs. per min. This moving disc produces a static torque on a parallel suspended disc of mica provided with a mirror, the steady deflection of which can then be measured. Pressures down to $10^{-6}\,\mu$ or less might thus be determined, given the appropriate mathematical constant, which, however, apparently remained to be calculated.

Dr. P. E. SHAW (communicated remarks) : Mr. Whipple's results obtained by the application of exact analysis will, I think, prove very useful. Each of the three cases (lamina, cylinder and sphere) is applicable to well-known special high vacuum apparatus. The pressure existing at any time in a sealed-off high vacuum cannot be found by the McLeod gauge ; the use of the electric discharge method is very rough, and may introduce ionisation, de-occlusion and other troubles. Other methods, no doubt, are possible ; but when the apparatus has a system moving with pendular or torsional vibration the log. dec. method is simple in the extreme, and requires no accessory plant. The operator need not go near the apparatus, for a mirror and telescope will show the amplitude of vibration. In the gravitation apparatus I used the amplitude could be detected accurately to an angle of 2 secs. The damping, in one case, for a vacuum of about $1/1,000,000$ atmosphere press. was about 3-4 secs. of angle This shows the sensitiveness obtainable. For special cases the method would prove convenient, and I think Mr. Whipple's investigation will prove useful to the experimentalist. The 20 per cent. reduction in the values of the pressure, as found by Mr. Whipple, does not surprise me ; for, as stated, I only aimed at approximate values.

XX. *An Alternating Current Bridge Method of Comparing Two Fixed Inductances at Commercial Frequencies. By* T. PARNELL, *M.A., Lecturer in Physics, University of Queensland.*

THE simple Maxwell inductance bridge cannot be used conveniently for the comparison of fixed inductances on account of the tedious double balance necessary, though by reason of its simplicity and symmetry the bridge has much to recommend it.

By making use of a current detector whose deflections depend on the component of the current in quadrature with the E.M F. applied to the bridge, it is possible to make the condition for no deflection depend either chiefly on the inductances or chiefly on the resistances.

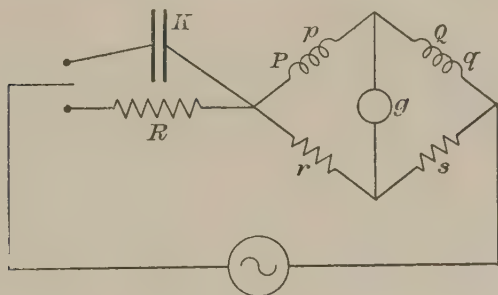The arrangement of the bridge is shown in Fig. 1. In series

FIG. 1.

with the bridge is placed either a non-inductive resistance, R, or a condenser, K. If R, r and s are all large compared with p, q, g, $P\omega$ and $Q\omega$, it will be shown that, with R in circuit, the balance depends chiefly on the inductances P and Q, and that with K in circuit the balance depends chiefly on the resistances.

With R in circuit, s is adjusted for no deflection ; this gives an approximate balance for the inductances. Then, with K in circuit, q is adjusted for no deflection ; this gives an approximate balance for the resistances. The process is repeated, and after a few balances values of s and q are obtained that

give a balance with either R or K in circuit, in which case the bridge is completely balanced, and

$$\frac{P}{Q}=\frac{p}{q}=\frac{r}{s}.$$

The first few balances need only be made roughly, and the whole process can be carried out very quickly; when the apparatus is adjusted, an unknown inductance can be balanced against a standard to one part in a few thousand in a few minutes.

As detector a sensitive moving-coil galvanometer and rotating commutator may be used. In this case the commutator is mounted on the axle of the small alternator or rotary converter that supplies the bridge current, and the brushes of the commutator are adjusted till no deflection is produced when the E.M.F. used on the bridge is applied through the commutator to a high non-inductive resistance in series with the shunted galvanometer, thus showing that the commutator is in quadrature with the E.M.F. When the commutator is thus adjusted the deflection of the galvanometer is approximately proportional to the component of the current in quadrature with the E.M.F. applied to the bridge.

A more convenient detector is the Sumpner reflecting electrodynamometer, manufactured by Paul. This instrument is fully described in a Paper by Sumpner and Phillips.* The suspended coil moves in the field of a laminated iron electromagnet excited by a field winding with a large number of turns; the resistance of this field winding is small compared with its reactance, and therefore the total flux through the field coils is approximately in quadrature with the applied E.M.F. The flux density is sensibly constant throughout the air-gap and proportional to the total flux. When, therefore, the moving coil is connected across the bridge, and an E.M.F. in phase with the E.M.F. used for the bridge is applied to the field coils, the mean torque on the moving coil is approximately proportional to the component of the current in quadrature with the bridge E.M.F.

If the mutual inductances between the various conductors, and also their capacities and the inductances of R, r and s, are negligible, the final balance is independent of wave-form and frequency. The preliminary balances depend on wave-form and frequency. An approximate knowledge only of the con-

* " Proc." Phys. Soc., Vol. XXII., p. 395. " Phil. Mag.," August, 1910, p. 309.

ditions for these preliminary balances is required; an approximate calculation of the current through the detector, therefore, is sufficient, and it may be assumed with a sufficient degree of accuracy that the mean torque on the moving coil is proportional to the component of the current in quadrature with the bridge E.M.F.
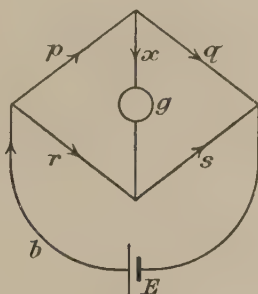

Fig. 2.

Consider the bridge as in Fig. 2

For continuous currents $x = \dfrac{qr - ps}{\Delta} \cdot E$,

where $\Delta = bg(p+q+r+s) + b(p+r)(q+s) + g(p+q)(r+s)$
$\qquad\qquad + pq(r+s) + rs(p+q)$.

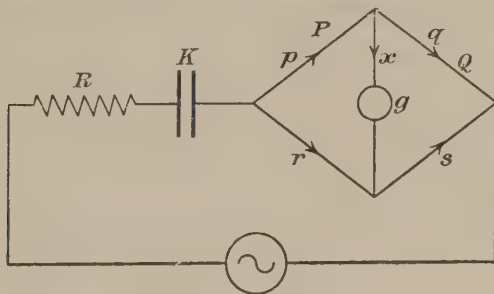For the bridge, Fig. 3, with the harmonic E.M.F., $e = E \sin \omega t$,

Fig. 3.

if the inductance of g is small compared with P and Q,

p becomes the operator $p + i\omega P$
q ,, ,, $q + i\omega Q$
b ,, ,, $R - \dfrac{i}{\omega K}$

X 2

and if R, r, s are large compared with p, q, g, $P\omega$ and $Q\omega$, Δ becomes

$$(p+i\omega P)\left(r+R-\frac{i}{\omega K}\right)s+(q+i\omega Q)\left(s+R-\frac{i}{\omega K}\right)r$$
$$+\left(R-\frac{i}{\omega K}\right)rs \text{ approximately}$$
$$=M+iN,$$

where $\quad M=p(rs+Rs)+q(rs+Rr)+Rrs+\dfrac{1}{K}(sP+rQ)$

$$M=\omega P(rs+Rs)+\omega Q(rs+Rr)-\frac{1}{\omega K}rs.$$

$$\therefore \quad x=\frac{(qr-ps)+i(\omega Qr-\omega Ps)}{M+iN}\ .\ E.$$

If $A=\dfrac{q}{p}-\dfrac{s}{r}$, i.e. the " resistance error," and $B=\dfrac{Q}{P}-\dfrac{s}{r}$, i.e. the " inductance error,"

$$x=\frac{prA+i\omega PrB}{M+iN}\ .\ E$$
$$=\frac{prMA+\omega PrNB+i(\omega PrMB-prNA)}{M^2+N^2}E.$$

The mean torque on the moving coil is proportional to the component of x in quadrature with E, i.e. to

$$\frac{\omega PrMB-prNA}{M^2+N^2}\ .$$

$$\therefore \quad \text{Mean torque on moving coil}=XB-YA$$

where $\quad \dfrac{X}{Y}=\dfrac{\omega PM}{pN}$

$$=\frac{\omega P\left\{p(rs+Rs)+q(rs+Rr)+Rrs+\frac{1}{K}(sP+rQ)\right\}}{p\left\{\omega P(rs+Rs)+\omega Q(rs+Rr)-\frac{1}{\omega K}\cdot rs\right\}}\ .$$

Consider now the two cases for preliminary balances :—

Case I. (inductive)—

$$\frac{1}{K}=0$$

$$\frac{X}{Y}=\frac{\omega PRrs}{\omega p\{P(rs+Rs)+Q(rs+Rr)\}}\quad \text{approx.}$$

$$=\frac{Rrs}{prs+pRs+\frac{Q}{P}(prs+pRr)}\ .$$

$$\therefore \quad \frac{X}{Y} \text{ is positive and fairly large.}$$

Case II. (resistance)—

$$R=0$$

$$\frac{X}{Y}=\frac{\omega P\left\{(p+q)rs+\frac{1}{K}(sP+rQ)\right\}}{p\left\{rs\omega(P+Q)-rs\frac{1}{\omega K}\right\}}$$

$$=-\frac{\omega^2 P\left\{(p+q)rs+\frac{1}{K}(sP+rQ)\right\}}{\frac{1}{K}prs}\quad\text{approx.}$$

$\therefore\ \dfrac{X}{Y}$ is negative, and small for ordinary values of P, Q and ω.

The above assumes an harmonic E.M.F., but since the final balance is independent of frequency and wave-form, the effect of the higher harmonics in the case of the E.M.F. obtained from an ordinary machine may be neglected, for it is clear from the above expressions that the effect of the higher harmonics in the preliminary balances will be much less than that of the fundamental.

A more rigorous treatment leads to the same result as the simpler theory given above.

For no deflection the mean torque is zero—

i.e. $\quad XB-YA=0$;

or $\quad X\left(\dfrac{Q}{P}-\dfrac{s}{r}\right)-Y\left(\dfrac{q}{p}-\dfrac{s}{r}\right)=0.$

If this preliminary balance is obtained by varying s with R alone in circuit (Case I.), X and Y are of the same sign, and X is fairly large compared with Y ; hence the balance depends chiefly on the inductances. If in this balance q is too small for a resistance balance A is negative, and therefore B is negative, and the value of s is too great. Similarly if the value of q is too great, the value of s obtained will be too small.

If a balance be now obtained by varying q (or p) with K alone in circuit (Case II.), X and Y are of opposite sign and X is fairly small compared with Y ; hence the balance depends chiefly on the resistances. If in the first balance q is too small, a value of s too great for a true inductive balance is obtained— *i.e.*, in this balance B is negative, and therefore A is positive, and the value of q is too great. The next inductive balance will, therefore, give too small a value of s, which means

that the next resistance balance will give too small a value of q, and so on.

Two series of values of s and q respectively are thus obtained whose terms are alternately too great and too small, and these terms rapidly converge to final values corresponding to a balance of both inductances and resistances—

i.e.
$$\frac{P}{Q} = \frac{r}{s} = \frac{p}{q}.$$

The following experimental results serve to illustrate the use of the method and to show the agreement of the above theory with experiment.

In these measurements P was a standard inductance and Q the coil to be measured. An inductive balance was first obtained by varying s, then a resistance balance by adding resistance to the arm containing Q, or to the arm containing P, by means of a dial box and graduated non-inductive rheostat, which could be switched into either arm.

Column I. gives the series of values of s for inductive balances; Column II. the corresponding values of the added resistances.

One example of the use of the rotating commutator is given; the other measurements were made with the electrodynamometer as detector.

Coil A.—Resistance, 15·6 ohms; $P = 100$ M.H., $R = 2,000$ ohms, $K = \frac{1}{2}$ M.F., $r = 2,000$ ohms, $p = 34·8$ ohms

Using electrodynamometer.		Using rotating commutator.	
I. s.	II. Resistance added to arm Q.	I. s.	II. Resistance added to arm Q.
3,480·0	0		0
3,040·0	45·0	3,300·0	43·0
3,120·0	37·0	3,060·0	37·0
3,099·0	38·7	3,108·0	38·22
3,102·6	38·39	3,101·0	38·04
3,102·3	38·425	3,102·0	38·06
3,102·4	38·42	3,101·7	...
...		...	
$Q = 155·12$ M.H.		$Q = 155·08$ M.H.	

Coil B.—Resistance, 0·73 ohm ; $P = 10$ M.H.. $R = 2\,000$ ohms, $K = \frac{1}{2}$ M.F., $r = 2,000$ ohms, $p = 7·4$ ohms.

I. *s.*	II. Resistance added to arm Q.
	0
1,000·0	
	3·0
800·0	
	2·2
860·0	
	2·45
846·0	
	2·36
853·0	
	2·405
850·0	
	2·39
850·1	
$Q = 4·250$ M.H.	

Coil C.—Resistance, 0·3 ohm ; $P = 10$ M.H., $R = 2\,000$ ohms, $K = \frac{1}{2}$ M.F., $r = 2,000$ ohms, $p = 7·4$ ohms.

I. *s.*	II. Resistance added to arm Q.
	0
129·0	
	0·13
119·0	
	0·09
121·4	
	0·105
120·0	
	0·101
120·6	
	0·102
120·5	
$Q = 0·602$ M.H.	

With the larger inductances the final balance could easily be made to 1 in 10,000, but this degree of accuracy cannot be expected in the actual results, as the non-inductive resistance boxes were specified as correct to 1 in 1,000 only and no small corrections were made.

Measurements of a number of coils with various values of the voltage and of R, K and r gave results consistent within 1 in 1,000. As a further test of the accuracy of the method. two coils were placed a considerable distance apart, and their in-

ductances (including the leads) measured in situ, singly and in series. The results of two such measurements are given below.

Inductance of A	155·12 M.H.
Inductance of A'	158·02 M.H.
Sum of inductances	313·14 M.H.
Inductance of A and A' in series	313·24 M.H.
Inductance of B	4·250 M.H.
Inductance of B'	3·942 M.H.
Sum of inductances	8·192 M.H.
Inductance of B and B' in series	8·190 M.H.

Reversing the connections of A' or B' did not appreciably affect the final balances.

Ow ng to the presence of R or K, it is necessary to use fairly high voltages in order to keep up the sens tiveness. The value of R need not be known with much accuracy, so that any reasonably non-inductive resistance may be used, and as and s are large compared with p and q, the voltage is limited only by the permissible current through the inductance coils. In the measurements made, voltages from 10 to 85 were used, and the higher voltages were found the more sat sfactory

The mutual induction between R and the rest of the bridge is liable to produce considerable error ; it is necessary, therefore, to keep R some little distance away.

Leakage to earth and electrostatic capacity effects gave some trouble. The detector was kept permanently in circuit, and the alternating-current supply connected to the two terminals of the bridge through a pair of tapping keys ; the apparatus was manipulated until putting down either key alone gave only a negligible deflection. In obtaining a balance the two keys were put down simultaneously. The " false zero " obtained by keeping the detector in circuit fluctuates to some extent, but little difficulty was experienced from this cause, as it was always sufficiently steady to enable the final balance to be made accurately.

When the rotating commutator was used it was found that the best resul s were obtained by taking the bridg current from a small transformer, with secondary well insulated and shielded from primary connected to the machine. The galvanometer was connected to the brushes bearing on the split part of the commutator, and the bridge to the brushes on the rings. When the core of the transformer, the frame of the machine and one of the rings of the commutator were

earthed, the stray effects due to leakage, capacity, &c., were reduced to a negligible amount.

In setting the brushes of the commutator the current from the transformer was passed through the commutator and the suitably shunted galvanometer in series with R, and he brushes adjusted for no deflection. This allowed for any change of phase occurring when the bridge current was taken from the transformer. It was found that great accuracy in the position of the brushes was not necessary. If the setting was not sufficiently exact, the defect was noticed at once, as the values of s and q did not alternate in the manner described above.

The use of the rotating commutator was found much more troublesome than that of the electrodynamometer, and the results obtained were somewhat less consistent.

When the electrodynamometer was used as detector, the currents for the bridge and for the field were taken from a small transformer having two secondaries well insulated and shielded from one another. It was found that with this arrangement electrostatic and capacity effects could be reduced to a negligible amount by earthing the core of the transformer and one terminal of the moving coil. Voltages from 40 to 75 volts were employed on the field coils.

In conclusion I wish to express my thanks to Prof. Lyle for his kindness in reading through this Paper and offering valuable criticisms. It was at his suggestion that the simpler proof given here was substituted for a more elaborate and cumbersome one employed originally.

NOTE.—Sumpner and Phillips mention the use of the electrodynamometer for the d rect comparison of inductances. but they state that the resistance balance is unnecessary This does not agree at all with the results obtained in this Paper.

ABSTRACT.

The Paper describes a method of avoiding the troublesome double adjustment required in Maxwell's method of comparing inductances. A current detector, whose deflections depend on the component of the current in quadrature with the E.M.F., is employed, which makes it possible to arrange that the condition for no deflection depends chiefly either on the inductances or resistances. In series with the bridge is placed either a non-inductive resistance or a capacity. In the first case the balance depends chiefly on the inductances, and in the second case on the resistances. A few alternate repetitions of the two adjustments suffice to balance the bridge, both for resistances and inductances. As detector a sensitive moving coil galvanometer

in conjunction with a commutator, or a Sumpner electrodynamometer, may be employed ; the latter proved more satisfactory.

DISCUSSION.

Dr. D. OWEN regarded the Paper as a valuable contribution to the subject of alternate-current bridge measurements. To secure zero current through the detector of a bridge to which an alternating voltage is applied it is necessary to satisfy two conditions. It has been claimed that by the use of the alternate-current moving-coil galvanometer inductances or capacities might be accurately determined without the necessity of a double adjustment. The claim is a mistaken one. The mere attainment of zero deflection, implying zero torque on the moving coil, by itself cannot give the information sought, for neither of the conditions for zero current in the detector is necessarily satisfied. The author of the present Paper shows a way out of the difficulty—namely, by applying alternately voltages to the bridge which are approximately—but not necessarily exactly—in quadrature with each other, the voltage applied to the field coil remaining unchanged. The attainment of a balance in *both* of these cases justifies the inference that the galvanometer current due to the voltage impressed on the bridge is zero, and that the two fundamental conditions of balance are simultaneously reached. A Paper by Weibel in the Bureau of Standards "Bulletin" (February, 1917) deals very fully with this same problem. Weibel points out how the tests may be made also by varying the phase of the voltage applied to the field coil, voltages in quadrature being secured by the aid of a phase-shifting device. The present author applies his method solely to the Maxwell bridge for comparing two self-inductions. There seems, however, no good reason for this exclusive treatment. The mode of procedure advocated is applicable to all the bridge methods, and those methods in which the two conditions of balance are independently attainable will still have the advantage in practice over those, such as the Maxwell bridge of the Paper, in which the opposite is the case.

Mr. T. SMITH pointed out that in double adjustments of this nature, if the wrong adjustment were made first, the successive errors got larger instead of smaller. It would have been helpful if the author had extended his mathematics somewhat, and given some criterion (*e.g.*, some function of the various quantities involved being either positive or negative) as to which of the adjustments—inductance or resistance—should be made first.

Dr. BRYAN asked if the method could be used for small inductances of a few micro-henries ?

Prof. BOYS asked if inductances with iron cores could be measured, or only air cored coils, in which the sine law was nearly fulfilled ?

Dr. SUMPNER said, in reply, that the author had not gone in great detail into the range of usefulness of the method. He might point out, however, that the sine law was not essential to work with the electrodynamometer. The author did not appear to have experienced any difficulty about the order of the adjustments ; though in some cases this might doubtless cause considerable trouble. He admitted the validity of the author's criticism of the Paper by himself and Phillips ; and agreed that the double adjustment was essential. At that time, however, he was principally concerned with describing the instrument, and was not using small inductances in which the resistance effect was important.

Mr. A. CAMPBELL communicated the following : Mr. Parnell's system is interesting, and may be of value in other methods where two adjustments are necessary to obtain a balance. In connection with his commutator method it may be mentioned that L. T. Robinson applied a commutator in the galvanometer circuit in a somewhat similar way for testing transformers by inductance methods ("Trans." American Institute of Electrical Engineers, p. 1040, Vol. XXVIII., June 30, 1909).

XXI. *On the Wave-lengths and Radiation of Loaded Antennæ.*
By BALTH. VAN DER POL, JUN., *D.Sc.* (*Utrecht*).

THE real function of an antenna as used in wireless tele-
graphy is to radiate electromagnetic waves into space. The
actual measurement of the amount of this radiation is a diffi-
cult matter, though some suggestions in this direction have
been made by Erskine-Murray * and De Groot.† To reach a
high degree of exactness in such measurements, however, seems
to be hardly possible as yet, so that a general theoretical
investigation may have some interest.

FIG. 1c.
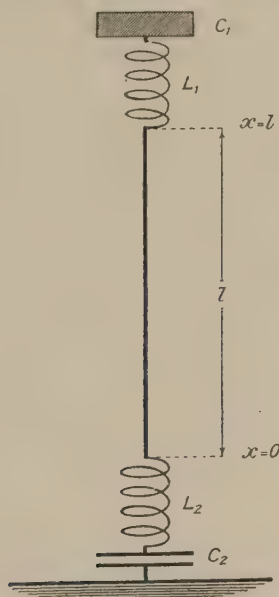
Antennæ in practice have a great variety of forms. They
often consist of one or more straight, parallel, vertical wires
continued at the top by others more or less horizontal. At the
bottom of the vertical part an inductance-coil is placed, used
as one side of a mutual inductance arrangement for the pur-
pose of transferring electromagnetic energy from a primary
oscillation circuit to the radiator.

* " Jahrbuch der Drahtl. Telegr. und Teleph.," V., 499, 1912.
† " Jahrbuch der Drahtl. Telegr. und Teleph.," VIII., 109, 1914.

A condenser of variable capacity is often inserted at the earthed base of the aerial, especially when the latter is used as a receiving arrangement. The horizontal wires at the top of a symmetrical antenna act principally as a capacity attached to the vertical part of the aerial.

To bring the many forms of these arrangements within the reach of mathematical analysis we assume that the antenna has the form of Fig 1.—*i.e.*, there is a long vertica part in which the static capacity and the static self-inductance per unit length ($\overline{C}$ and $\overline{L}$) are equally distributed over the whole length. At the same time a self-inductance L_2 and a condenser of capacity C_2 are inserted at the earthed bottom end of the antenna, and at the top we suppose a self-inductance L_1 and capacity C_1 attached in addition. The inductance L_1 is generally not found in actual antennæ, but is inserted here to make the problem symmetrical.

The first things to find are the distributions of the current and of the potential over the vertical part, and also the possible wave lengths with which the whole system can vibrate.*

Taking the vertical as axis of x, with the zero point at the bottom of the vertical wire (*see* Fig. 1), whose length is l; calling the current at any part i and the potential v, we have to find expressions for i and v which will be a solution of the equations,

$$\left.\begin{aligned}\overline{L}\frac{di}{dt}&=-\frac{dv}{dx}\\[2mm]\overline{C}\frac{dv}{dt}&=-\frac{di}{dx}\end{aligned}\right\}, \quad \cdots \quad \cdots \quad (1)$$

with the boundary conditions,

$$\text{for } x=o: \qquad v=-\frac{1}{C_2}\int_0^t i\,dt-L_2\frac{di}{dt}$$
$$\text{and for } x=l: \qquad v=\frac{1}{C_1}\int_0^t i\,dt+L_1\frac{di}{dt} \qquad \left.\right\}, \quad \cdots \quad \cdots \quad (2)$$

taking the direction of x from the bottom to the top as positive.

In order to simplify the problem, we assume that undamped oscillations exist in the antenna. This assumption is valid, as the distribution of the current is, as a first approximation, independent of the damping.

* With a loading at the bottom, but none at the top of the aerial the values of the possible wave-lengths are found by L. Cohen, " Electrical World," LXV., 286, 1915 ; and W. H. Eccles, " Handbook of Wireless Telegr. and Teleph.," p. 121.

· Combining the two equations (1) we are led to the well-known formula,

$$LC\frac{d^2i}{dt} = \frac{d^2i}{dx^2}, \quad \cdots \quad \cdots \quad (3)$$

and a similar equation in v.

If we call the velocity of the disturbance $a = 1/\sqrt{CL}$, the angular frequency of the vibration ω, and take a suitable zero-point of time, the solution of (3) for one single frequency may be written in the form,

$$\left.\begin{aligned} i &= T_0 \cos\left(\frac{\omega x}{a} - \beta\right).\cos\omega t \\ \text{and} \qquad v &= V_0 \sin\left(\frac{\omega x}{a} - \beta\right).\sin\omega t \end{aligned}\right\}, \quad \cdots \quad (4)$$

where T_0 and $V_0 = \sqrt{(L/C)}$. T_0 are the maximum current and potential amplitudes respectively at an antinode. This antinode eventually may not be found on the antenna itself, but can always be calculated when the actual current distribution over the antenna is known.

The quantities ω and β are to be found by aid of the conditions (2).

Putting $\qquad\qquad \theta = \omega l/a = 2\pi l/\lambda, \quad \cdots \quad \cdots \quad (5)$

the total vertical capacity

$$C = Cl,$$

and the total vertical inductance

$$L = Ll,$$

then applying (2) to (4) the following expressions are easily deduced :—

$$tg\beta = \left(\frac{1}{C_2\omega} - L_2\omega\right)\sqrt{\frac{C}{L}} = \frac{C}{C_2\theta} - \frac{L_2\theta}{L}, \quad \cdots \quad (6)$$

and $\qquad tg\left(\frac{\omega l}{a} - \beta\right) = \left(\frac{1}{C_1\omega} - L_1\omega\right)\sqrt{\frac{C}{L}} = \frac{C}{C_1\theta} - \frac{L_1\theta}{L}. \quad \cdots \quad (7)$

Now, putting

$$\frac{\omega l}{a} - \beta = \theta - \beta = a, \quad \cdots \quad \cdots \quad (8)$$

we have $\qquad\qquad tga = \frac{C}{C_1\theta} - \frac{L_1\theta}{L}, \quad \cdots \quad \cdots \quad (9)$

and we get the following transcendental equation—

$$tg\theta = \frac{tga + tg\beta}{1 - tga\, tg\beta} = \frac{\left(\frac{C}{C_1\theta} - \frac{L_1\theta}{L}\right) + \left(\frac{C}{C_2\theta} - \frac{L_2\theta}{L}\right)}{1 - \left(\frac{C}{C_1\theta} - \frac{L_1\theta}{L}\right)\left(\frac{C}{C_2\theta} - \frac{L_2\theta}{L}\right)}. \quad \cdots \quad (10)$$

There are an infinite number of solutions θ to this equation, each one of which corresponds to a natural wave-length of the loaded antenna according to·(5) and they can be found when the ratios C/C_1, L/L_1, C/C_2 and L/L_2 are known. The values of θ which satisfy (10) are best found by determining the points of intersection of a curve representing the periodic function $tg\theta$, and the function given by the second member of (10)

When using formula (10) it should be remembered that if there is no loading either at the top or bottom of the antenna C_1, L_1 and L_2 are each to be taken as zero, but C_2 must be taken as infinite.

From (4) and (8) it is clear that α and β have the following physical meanings :—The current and potential at the bottom of the antenna have maximum amplitudes respectively of $T_0 \cos \beta$ and $V_0 \sin \beta$, and the maximum current and potential at the top are $T_0 \cos \alpha$ and $V_0 \sin \alpha$.

We can find the values of θ, α and β experimentally by first measuring the wave-length of an antenna when the load at the top in the form of horizontal wires is present, but with no loads at the bottom. The height l being known (5) gives us θ. For no bottom load we have to take, according to (6), $C_2 = \infty$, $L_2 = 0$; which gives $\beta = 0$, so that $\alpha = \theta$. In this case we have, therefore,

$$tg\theta = C/C_1\theta,$$

from which C/C_1 can be found. When, now, any capacity and or) inductance is inserted at the bottom, leaving the top unaltered, and the wave-length λ' is measured again, we have

$$tg\alpha' = C/C_1\theta',$$

from which α' can be calculated, C/C_1 and θ' being known. Again, β' is found by the subtraction $\theta' - \alpha' = \beta'$.

Formula (10) shows us that it is possible, by varying the loads, to have a constant wave-length with different distributions of the potential and current over the antenna, which results in bringing the same sine or cosine function at different altitudes on the antenna.

This is of much importance in c nnection with the radiation that is s nt off the antenna, as will be seen later on.

To find the *total antenna energy E* we have to compute the value of

$$E = \tfrac{1}{2}\bar{L}\int_0^l i^2 dx + \tfrac{1}{2}L_2 i^2_{x=0} + \tfrac{1}{2}L_1 i^2_{x=l} \quad . \quad . \quad . \quad (11)$$

at the time of maximum current, or, what is the same thing, the value of

$$E = \tfrac{1}{2}\bar{C}\int_0^l v^2 dx + \tfrac{1}{2}C_2 v_2{}^2 + \tfrac{1}{2}C_1 v_1{}^2 \quad . \quad . \quad . \quad (12)$$

at the time of maximum potential, v_1 and v_2 being the values of the potential at the top and bottom capacities respectively.

Now (11) is equal to—

$$E - \tfrac{1}{4}T_0{}^2 L \left\{ 1 + \frac{1}{2\theta}(\sin 2\alpha + \sin 2\beta) \right\} + \tfrac{1}{2}T_0{}^2 L_1 \cos^2\alpha + \tfrac{1}{2}T_0{}^2 L_2 \cos^2\beta,$$

and from (12) we get $\qquad\qquad . \quad . \quad (13)$

$$E - \tfrac{1}{4}T_0{}^2 L \left\{ 1 - \frac{1}{2\theta}(\sin 2\alpha + \sin 2\beta) \right\} + \frac{1}{2\theta^2}T_0{}^2 \frac{LC}{C_1}\cos^2\alpha$$
$$+ \frac{1}{2\theta^2}T_0{}^2\frac{LC}{C_2}\cos^2\beta, \quad (14)$$

which can be shown to be the same.

Taking the mean of (13) and (14) we get the more symmetrical expression for the total energy of the loaded antenna :

$$E = \tfrac{1}{4}T_0{}^2 L\left[1 + \frac{\cos^2\alpha}{\theta}\left(\frac{L_1\theta}{L} + \frac{C}{C_1\theta}\right) + \frac{\cos^2\beta}{\theta}\left(\frac{L_2\theta}{L} + \frac{C}{C_2\theta}\right) \right] \quad . \quad (15)$$

The quotient of the total electromagnetic antenna-energy by half the square of the maximum current at a current antinode may be called the *dynamic self inductance* L' of the complete system. In a similar way the *dynamic capacity* C' of the total system may be defined. It is clear that both L' and C' vary not only with the loads, but also with the harmonic in which the antenna is vibrating. The energy of an earthed vertical rod (plain Marconi aerial) vibrating in its fundamental wave-length ($\lambda = 4l$) or any higher harmonic is well known as being $\tfrac{1}{4}T_0{}^2 L$. As all terms in (15) are essentially positive we see that the energy of a loaded antenna, with the same maximum current value at an antinode, and also the dynamic self-inductance L' are always greater than the corresponding values of the same antenna when unloaded.

As we shall see further on, the radiation depends on the current distribution over the vertical part of the antenna. The original energy stored on the aerial can be increased (*see* (15), by increasing the dynamic self-inductance of the antenna. This can be done, *e.g.*, by increasing the self-inductance L_2 at the bottom. The wave-length and the values of the current over the aerial can then be kept constant by decreasing at the

same time the capacity of the condenser C_2 at the bottom in such a way that

$$tg\beta = \frac{C}{C_2\theta} - \frac{L_2\theta}{L}$$

is a constant.

By inserting, therefore, a greater inductance at the bottom and at the same time a smaller capacity in the arrangement of Fig. 1, keeping the wave-length constant we are able to diminish the radiating decrement of an antenna with the well-known result of sharper tuning.

The current distribution over the antenna now being known, we can proceed to find the *radiation* that will be thrown off the radiator.

In calculating this radiation we make the following assumptions :—

1. The vertical dimensions of the loading are small in comparison with the total height o the antenna as is actually the
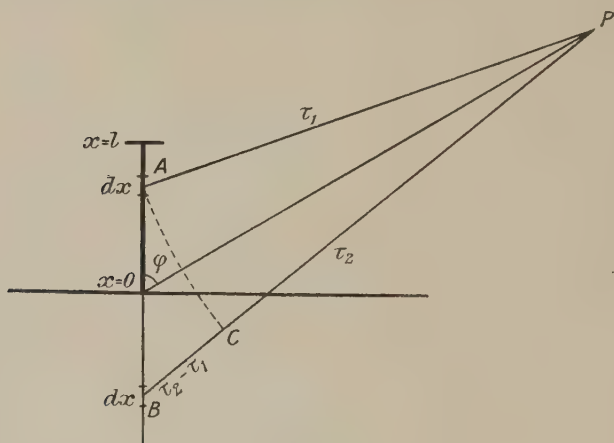
Fig. 2.

case, so that the contribution to the total radiation from the loadings are negligibly small.

2 The antenna is placed on a perfectly conducting earth, with the consequence that the lines of electric force round the radiator may be regarded as ending perpendicularly on the earth.

The electromagnetic field round the antenna may then be found, as shown by Abraham,[*] by using the electric image of the antenna with respect to the earth (see Fig. 2, in which AB is the antenna). Taking, now, a point P at a distance from the radiator great in comparison with the wave-length, we can find the intensity of the magnetic and electric fields due to the current in a part dx of the antenna, *and* a part dx of its image. According to Hertz,[†] the electric force is

$$dE = \frac{2\pi c}{\lambda r_1} i_x dx \sin\varphi \cdot \sin\left(\omega t - \frac{2\pi r_1}{\lambda}\right) + \frac{2\pi c}{\lambda r_2} i_x dx \sin\varphi \sin\left(\omega t - \frac{2\pi c_2}{\lambda}\right)$$

$$= \frac{4\pi c}{\lambda r} i_x dx \sin\varphi \cdot \sin\left(\omega - \frac{2\pi r}{\lambda}\right) \cdot \cos\frac{\pi(r_2 - r_1)}{\lambda}, \quad (16)$$

where we take $\dfrac{r_1 + r_2}{2} = r =$ the distance of P from the bottom of the antenna and where φ is the angle which the radius-vector to P makes with all parts of the antenna. It is clear that the disturbance at P coming from the antenna part dx will arrive there in a different phase from that of the disturbance originating at the same time from the image-part dx.

The magnetic field at P, due to the same sections of the radiator is

$$dH = \frac{4\pi}{\lambda r} i_x dx \sin\varphi \cdot \sin\left(\omega t - \frac{2\pi r}{\lambda}\right) \cdot \cos\frac{\pi(r_2 - r_1)}{\lambda}. \quad . \quad . \quad (17)$$

Now,

$$\frac{\pi(r_2 - r_1)}{\lambda} = 2\pi \frac{x}{\lambda} \cos\varphi \ (see \text{ Fig. 2, where } r_2 - r_1 = BC),$$

so that the electric and magnetic fields at P due to the whole antenna are

$$E = \frac{4\pi \cdot T_0}{\lambda r} \sin\varphi \cdot \sin\left(\omega t - \frac{2\pi r}{\lambda}\right) \int_{x=0}^{x=l} \frac{i_x}{T_0} \cos\left(\frac{2\pi x \cos\varphi}{\lambda}\right) dx, \quad (18)$$

and

$$H = \frac{4\pi T_0}{\lambda r} \sin\varphi \cdot \sin\left(\omega t - \frac{2\pi r}{\lambda}\right) \int_{x=0}^{x=l} \frac{i_x}{T_0} \cos\left(\frac{2\pi x \cos\varphi}{\lambda}\right) dx. \quad (19)$$

As the electric force is everywhere at right angles to the

[*] " Physikalische Zeitschrift," II., 329, 1901.
[†] " Ausbreitung der Electr. Kraft."

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magnetic force, we find the radiation through a unit surface perpendicular to the radius-vector in a time dt equal to

$$d\Sigma = \frac{1}{4\pi}EH\,dt = \frac{4\pi c T_0^2 \sin^2\varphi}{\lambda^2 r^2}\,.\,\sin^2\left(\omega t - \frac{2\pi r}{\lambda}\right)$$
$$\left[\int_0^l \frac{i_x}{T_0}\cos\left(\frac{2\pi x \cos\varphi}{\lambda}\right)dx\right]^2 dt. \quad . \quad (20)$$

Now, we found

$$i_x = T_0 \cos\left(\frac{\omega x}{a} - \beta\right)\cos\,\omega',$$

so that the integral in (20) has the value

$$\int_0^l \cos\left(\frac{\omega x}{a} - \beta\right).\cos\left(\frac{2\pi x \cos\varphi}{\lambda}\right)dx =$$
$$\int_0^l \cos\left(\frac{\theta x}{l} - \beta\right).\cos\left(\frac{\theta x}{l}\cos\varphi\right)dx =$$
$$\frac{l}{\theta \sin^2\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin(\theta-\beta) - \cos\varphi.\sin(\theta\cos\varphi).$$
$$\cos(\theta-\beta)\} =$$
$$\frac{l}{\theta \sin^2\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin\alpha - \cos\varphi.\sin(\theta\cos\varphi).\cos\alpha\}.$$

Hence, the mean radiation through the above-mentioned unit surface is

$$d\Sigma = \frac{2\pi c T_0^2}{\lambda^2 r^2}.\frac{l^2}{\theta^2 \sin^2\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin\alpha - \cos\varphi.\sin$$
$$(\theta\cos\varphi).\cos\alpha\}^2$$
$$= \frac{c T_0^2}{2\pi r^2 \sin^2\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin\alpha - \cos\varphi.\sin(\theta\cos\varphi).$$
$$\cos\alpha\}^2. \quad . \quad (21)$$

Integrating this expression over a sphere of radius r, and taking half of the result, as radiation exists only over the upper half of the sphere, we get as the expression for the total radiation*,

$$\Sigma = \frac{c T_0^2}{2}\int_0^\pi \frac{1}{\sin\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin\alpha - \cos\varphi.\sin(\theta\cos\varphi).$$
$$\cos\alpha\}^2 d\varphi$$
$$= c T^2\int_0^\pi \frac{1}{\sin\varphi}\{\sin\beta + \cos(\theta\cos\varphi).\sin\alpha - \cos\varphi.\sin(\theta\cos\varphi).$$
$$\cos\alpha\}^2 d\varphi, \quad . \quad (22)$$

where T is the R.M.S. current at a current antinode.

* It is clear that the integral given by Abraham in "Phys. Zeitschr.," II., 332, 1901, for the total radiation of a plain rod must be $\int_0^\pi$ instead of $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$.

Putting R_ρ for the radiation resistance, *i.e.*, the quotient of the total radiation by the *M.S. current in an antinode*, we have

$$\Sigma = T^2 R_\rho^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (22\text{A})$$

so that

$$R_\rho = c \int_0^\pi \frac{1}{\sin \varphi} \{\sin \beta + \cos (\theta \cos \varphi) . \sin \alpha - \cos \varphi . \sin (\theta \cos \varphi) . \\ \cos \alpha\}^2 d\varphi.$$

This expression gives the radiation resistance in electromagnetic units. In ohms it is

$$R_\rho = 30 \int_0^\pi \frac{1}{\sin \varphi} \{\sin \beta + \cos (\theta \cos \varphi) . \sin \alpha - \cos \varphi . \sin (\theta \cos \varphi) . \\ \cos \alpha\}^2 d\varphi. \quad . \quad (23)$$

To evaluate the integral (call it I) in (23), we can proceed as follows

$$I = \int_0^\pi \frac{1}{\sin \varphi} \{\sin \beta + \cos (\theta \cos \varphi) . \sin \alpha - \cos \varphi . \sin (\theta \cos \varphi) . \\ \cos \alpha\}^2 d\varphi$$

$$= \int_{-1}^{+1} \frac{1}{1-u^2} \{\sin \beta + \cos \theta u . \sin \alpha - u \sin \theta u . \cos \alpha\}^2 du$$

$$= \tfrac{1}{2} \int_{-1}^{+1} \left(\frac{1}{1+u} + \frac{1}{1-u}\right) \{\sin \beta + \cos \theta u . \sin \alpha - u \sin \theta u . \\ \cos \alpha\}^2 du.$$

It is easy to show that the integral with the factor $\frac{1}{1+u}$ is equal to that with $\frac{1}{1-u}$ as factor, so that

$$I = \int_{-1}^{+1} \frac{1}{1+u} \{\sin \beta + \cos \theta u . \sin \alpha - u \sin \theta u . \cos \alpha\}^2 du$$

$$= \int_0^2 \frac{1}{x} \{\sin \beta + \cos \theta (x-1) . \sin \alpha - (x-1) \sin \theta (x-1) . \\ \cos \alpha\}^2 dx \quad . \quad (24)$$

Squaring the function under the integral sign and separating the terms with factors x^{-1}, x^0 and x^{+1}, we can resolve

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* The M.S. current *at the bottom* of the antenna in the definition for the radiation resistance is generally used instead of the M.S. current in an antinode. When, however, a condenser of very small capacity is inserted the current at the bottom of the antenna is zero and the radiation resistance becomes infinite, so that the total radiation in that case cannot simply be expressed as a function of the radiation resistance. It is for that reason that the radiation resistance is defined here in relation to the current amplitude in an antinode. The current at the bottom being $T_0 \cos \beta$, it is easy to convert the values found here to the more usual expression.

the complete integral I into three separate integrals I_1, I_2 and I_3, of which

$$I_1 = \sin 2\beta \int_0^2 \frac{\sin \theta x}{x} dx + (1 - \cos 2\beta) \int_0^2 \frac{1 - \cos \theta x}{x} dx$$

$$- \tfrac{1}{2} \sin 2\beta \int_0^2 \frac{\sin 2\theta x}{x} dx + \tfrac{1}{2} \cos 2\beta \int_0^2 \frac{1 - \cos 2\theta x}{x} dx$$

$$= \sin 2\beta \int_0^{2\theta} \frac{\sin \varphi}{\varphi} d\varphi + (1 - \cos 2\beta) \int_0^{2\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi$$

$$- \tfrac{1}{2} \sin 2\beta \int_0^{1\theta} \frac{\sin \varphi}{\varphi} d\varphi + \tfrac{1}{2} \cos 2\beta \int_0^{4\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi$$

$$= \sin 2\beta \,.\, Si\, 2\theta - \tfrac{1}{2} \sin 2\beta \,.\, Si\, 4\theta + (1 - \cos 2\beta) \int_0^{2\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi$$

$$+ \tfrac{1}{2} \cos 2\beta \int_0^{\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi \quad . \quad . \quad (25)$$

When using $\theta = \alpha + \beta$, I_2 (with the factor x^0) can be brought into the form,

$$I_2 = \int_0^2 \Big[- \cos^2 \alpha + \cos \alpha \,.\, \cos \{2\theta(x-1) + \alpha\} - 2 \sin \beta \,.\, \cos \alpha$$

$$\int_0^2 \sin \theta(x-1) \Big] dx$$

$$= - \cos^2 \alpha \int_0^2 dx + \frac{\cos \alpha}{2\theta} \int_0^{4\theta} \cos (\varphi - \theta - \beta) d\varphi - \frac{2 \sin \beta \,.\, \cos \alpha}{\theta} \int_0^{2\theta}$$

$$\sin (\varphi - \theta) d\varphi$$

$$= - 2 \cos^2 \alpha + \frac{\cos \alpha}{2\theta} \{\sin (3\theta - \beta) + \sin (\theta + \beta)\} + 0$$

$$= \cos^2 \alpha \left(-2 + \frac{\sin 2\theta}{\theta} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

I_3 (with the factor x) is

$$I_3 = \cos^2 \alpha \int_0^2 x \,.\, \sin^2 \theta\, (x-1) dx = \cos^2 \alpha \left(1 - \frac{\sin 2}{2\theta} \right). \quad . \quad (27)$$

Therefore, we find as the complete integral—

$$I = I_1 + I_2 + I_3 = \sin 2\beta \,.\, Si\, 2\theta - \tfrac{1}{2} \sin 2\beta \,.\, Si\, 4\theta$$

$$+ (1 - \cos 2\beta) \int_0^{2\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi + \tfrac{1}{2} \cos 2\beta \int_0^{4\theta} \frac{1 - \cos \varphi}{\varphi} d\varphi$$

$$+ \cos^2 \alpha \left(\frac{\sin 2\theta}{2\theta} - 1 \right). \quad . \quad . \quad . \quad (28)$$

For small values of θ (i.e., for long wave-lengths) the integrals still occurring in (28) can be expanded in a rapidly converging series as follows :—

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi = \int_0^x \frac{1-1+\frac{\varphi^2}{2'}-\frac{\varphi^4}{4'}+-\cdots}{\varphi}d\varphi = \frac{x^2}{2\cdot 2!}+\frac{x^4}{4\cdot 4!}-\frac{x^6}{6\cdot 6!}$$
$$+\cdots$$

On the other hand, using the functions integral-cosines we have, when δ is a quantity small in comparison with 1,

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi = \int_0^\delta \frac{1-\cos\varphi}{\varphi}d\varphi + \int_\delta^x \frac{1-\cos\varphi}{\varphi}d\varphi$$

$$= \int_0^\delta \frac{2\sin^2\frac{1}{2}\varphi}{\varphi}d\varphi + \int_\delta^x \frac{d\varphi}{\varphi} - \int_\delta^x \frac{\cos\varphi}{\varphi}d\varphi$$

$$= \frac{1}{2}\int_0^\delta \varphi d\varphi + lg_e x - lg_e \delta - \int_\delta^\infty \frac{\cos\varphi}{\varphi}d\varphi + \int_x^\infty \frac{\cos\varphi}{\varphi}d\varphi$$

$$= \frac{1}{4}\delta^2 - lg_e\delta + lg_e x + Ci\,\delta - Ci\,x.$$

Letting δ approach zero, the limit of this expression becomes

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi = E + lg_e x - Ci\,x,$$

where E is Euler's constant, and is equal to
$$E = 0\cdot577216\ldots$$

Applying these results to (28) we can write

$$I = \frac{\sin 2\beta}{2}\{2\,Si\,2\theta - Si\,4\theta\} + \frac{\cos 2\beta}{2}\{2\,Ci\,2\theta - Ci4\theta - E - lg.\theta\}$$

$$+ E + lg.2 + lg_e\,\theta - Ci\,2\theta + \cos^2 a\left(\frac{\sin 2\theta}{2\theta}-1\right). \quad (29)$$

And we find as the general expression for the total radiation resistance of a loaded antenna, expressed in ohms * :—

$$R_\rho = 15\sin 2\beta\{2Si\,2\theta - Si\,4\theta\} + 15\cos 2\beta\{2Ci\,2\theta - Ci\,4\theta - lg.\theta$$

$$-0\cdot577216\} + 30\left\{\cos^2 a\left(\frac{\sin 2\theta}{2\theta}-1\right) - Ci2\theta + lg_e\theta + 1.27036\right\}$$

$$(30)$$

where $\theta = \frac{2\pi l}{\lambda}$ and $\cos a$ and $\cos\beta$ are the ratios of the current amplitudes at the top and bottom respectively of the antenna to the maximum current amplitude at an antinode.

* Tables of the integral-sine and integral-cosine functions are calculated by J. W. L. Glaisher, and published in the Philosophical "Transactions," London, CLX., 367, 1870.

Formula (30) shows us, as was to be expected, that the radiation-resistance of a loaded antenna and also the radiation from that antenna are dependent not only on the length of the waves which the aerial emits, but a so in a high degree on the value of the current at the bottom and top, or, in other words, on how the current at that certain wave-length is distributed over the aerial. It is therefore clear that the radiation from an antenna *cannot* be written in the simple form,

$$\Sigma = AT^2 \frac{l^2}{\lambda^2},$$

where A is a constant, and T the R.M.S. current at the base of the antenna, as is usually done in most text-books on wireless telegraphy * for it all depends on the manner in which the loading is varied (either at the top or the bottom, or both) to alter the wave-length λ. Even if we assume a flat top antenna—*i.e.*, one with a capacity load at the top in the form of horizontal wires (ship antenna and umbrella antenna)—the total radiation cannot be written as

$$\Sigma = 160\pi^2 T^2 \frac{l^2}{\lambda^2},$$

as is done by Austin, and on which basis he calculated a table of radiation resistances for flat-top antennæ.†

For a flat top antenna without any load at the bottom $(tg\beta = 0)$ we have from (30),

$$R_\rho = 57 \cdot 0208 + 34 \cdot 5288 \, lg_{10} \frac{l}{\lambda} - 15 Ci \, 4\theta + 30 \cos^2\theta \left(\frac{\sin 2\theta}{2\theta} - 1 \right) (31)$$

It is apparent that the use of the formula—

$$R_\rho = 160\pi^2 \frac{l^2}{\lambda^2}$$

to calculate tables of radiation resistances for flat-top antennæ up to ratios $\frac{l}{\lambda} = 0 \cdot 4$, as done by Austin, is an extrapolation of Rüdenberg's results ‡ that is not justifiable, so that the figures given there often differ very much from the values calculated with the aid of our formulæ. Austin does not even state clearly how the variations of λ are to be obtained by varying the top or the bottom loads.

* *See, e.g.*, Fleming, "The Principles of Electric Wave Telegraphy and Telephony," 3rd edition, p. 459.

† " Journal of the Washington Acad. of Science," Vol. I., No. 7, Nov., 1911. Reprinted in Fleming, "The Principles of Electr. Wave Telegr.," p. 460, and " Jahrbuch der Drahtl. Telegr. u. Teleph.," V., 417, 1912.

‡ Rüdenberg, " Ann. der Phys.," XXV., 446, 1908.

Rüdenberg's formula is valid only for the case of a very long wave-length emitted from a flat top or umbrella antenna with a capacity at the top large in comparison with the total capacity of the vertical part of the aerial, for then it is possible to deduce the following approximation for the radiation. For no load at the bottom $\beta=0$, and, therefore, $\theta=\alpha$. For a value of λ *large in comparison with the total antenna height* l, θ is, according to (5) a small quantity. Now, the radiation as given by (22A) and (30) is a function of θ only, namely—

$$\Sigma = T^2 \left\{ 15E + 30 \, lg.2 + 15 \, lg_e\theta - 15 \, Ci \, 4\theta + 30 \cos^2\theta \left(\frac{\sin 2\theta}{2\theta} - 1 \right) \right\}$$

Expanding the functions of θ occurring in this expression in a series of powers of θ, bearing in mind also that fo $\theta < 1$:—

$$Ci \, \theta = E + lg_e\theta - \frac{\theta^2}{2 \cdot 2!} + \frac{\theta^4}{4 \cdot 4!} - \frac{\theta^6}{6 \cdot 6!} + \ldots,$$

we find :

$$\Sigma = T^2 \{ 40 \, \theta^2 - 16 \, \theta^4 + \frac{200}{63} \theta^6 \ldots \}.$$

Taking only the first term, we have

$$\Sigma = 40 \, \theta^2 \, T^2 = 160 \, \pi^2 \, \frac{l^2}{\lambda^2} \, T^2,$$

which is the above-mentioned Rüdenberg's formula and which is only valid for wave-lengths great in comparison with the antenna height.

From (30) and (22A) it follows that *for a constant wave-length* the radiation from an antenna with different loadings at the top and at the bottom, can be expressed in the following way :—

$$\Sigma = T^2 \{ A_1 \sin 2\beta + A_2 \cos 2\beta + A_3 \},$$
$$\theta = \text{constant}$$

where A_1, A_2, and A_3 are constants for a certain ratio of wave-length and antenna height. Bearing in mind that the current distribution is given by

$$i_x = T_0 \cos \left(\frac{\omega x}{a} - \beta \right),$$

and taking for x that height on the antenna where a current-node is found, the total energy thrown off the antenna in the form of electromagnetic waves of a constant wave-length, but at different loads, is,

$$\Sigma = T^2 \left\{ B_1 \sin \frac{4\pi x}{\lambda} + B_2 \sin \frac{4\pi x}{\lambda} + B_3 \right\},$$

where B_1, B_2 and B_3 are constants.

To show the form of this function for two wave-lengths Fig. 3 has been calculated, in which the abscissa gives that part of the height of the antenna at which a current node occurs and the ordinate represents the antenna radiation-resistance at this current distribution, (1) for $\lambda=2l$, and (2) for $\lambda=4l$.

The formula for the radiation resistance of an antenna vibrating in a half wave-length—e g. when x is the height of the current-node, is—

$$R_{\rho}_{\lambda=2l}=58\cdot28-46\cdot32 \sin \left(\frac{2\pi x}{l}-1\cdot113\right)\Omega,$$

so that the radiation-resistance for this wave-length is a minimum and equal to $11\cdot96$ Ω when the current node occurs
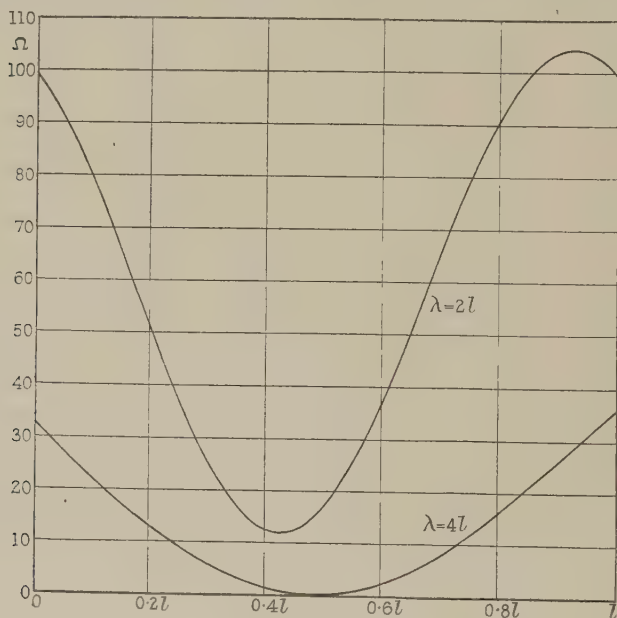
Fig. 3.

at a point $42\cdot7$ per cent. of the height of the antenna from the base, and the same is a maximum with a value of $104\cdot6$ Ω with the node at a height of $92\cdot7$ per cent. This example clearly illustrates how the current distribution, even by a constant wave-length, is of importance with regard to the radiation.

In the same Fig. 3 the curve for $\lambda=4l$ shows that there is a different radiation when an antenna vibrates in a quarter wave-length, but

(1) With the current node at the top and antinode at the bottom (plain Marconi aerial), or

(2) Reversed, with the antinode at the top and the node at the bottom, as can be approximated to by putting a large capacity at the top (umbrella antenna) and a very small capacity at the bottom.

In the first instance the radiation resistance is 36·6 Ω,* in the second (according to (30)) only 32·2 Ω.†

It is well known that the usual theory which does not take into account the phase difference at which the waves from various parts of the aerial reach a certain point in space, gives for both the cases just considered the value 40 Ω, a difference of 9·3 per cent. and 21·3 per cent. respectively from the more accurate formula.

It is interesting to find out in what directions the calculated electromagnetic energy is emitted from the antenna. This is best represented graphically. When we write the function under the integral sign in (22) in the form,

$$\frac{1}{\sin \varphi} \{\sin \beta + \cos(\theta \cos \varphi) . \sin \alpha - \cos \varphi . \sin (\theta \cos \varphi) . \cos \alpha\}^2$$

$$= \frac{1}{\sin \varphi} \{f(\varphi)\}^2,$$

polar curves in a plane can be drawn in four different ways having the following peculiarities and formulæ :—

1. If the radius vector r is made proportional to the energy radiated through a unit surface of a sphere of a constant radius described round the sending antenna, the curve has the equation,

$$r = \{f(\varphi)\}^2 / \sin^2 \varphi \quad . \quad . \quad . \quad . \quad (32)$$

2. If r is made proportional to the total energy radiated through that zone of a sphere that is comprised between two cones with a slightly different apex-angle, the formula is

$$r = \{f(\varphi)\}^2 / \sin \varphi. \quad . \quad . \quad . \quad . \quad (33)$$

3. r can be made proportional to that distance from the antenna where the radiation through a unit-surface, held at right angles to the radius vector, is a constant, or, in other

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* In accordance with Abraham. *See* Paper cited.

† The usual definition of radiation resistance would give here $R_\rho = \infty$.

words, r can be made proportional to that distance from the antenna where, by a given receiving arrangement, the amount of energy received is a constant. We thus have the formula,

$$r = f(\varphi)/\sin \varphi. \qquad \ldots \ldots \ldots \quad (34)$$

This surface has the peculiarity that the magnetic and electric forces are constant all over it. This curve can be called the characteristic curve of radiations (analogous to the similar curve in a horizontal plane for directive antennæ such as used by Bellini and Tosi), and represents at the same time the maximum distances in different directions where signals can be received.

4. If

$$r = f(\varphi)/\sqrt{\sin \varphi}, \qquad \ldots \ldots \ldots \quad (35)$$

the area of this surface will be proportional to the total energy radiated.

The accompanying curves are calculated from formula (33), as these curves are apparently the most suitable ones for practical design.
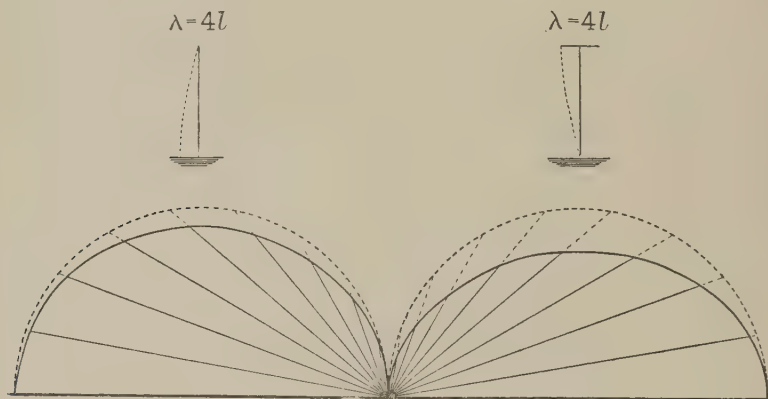
Fig. 4

In Fig. 4 the solid line at the left represents the distances in all directions in a vertical plane from a simple rod-aerial, earthed at the bottom, where the received energy will be constant. The fundamental wave-length is thus equal to $4l$, and the current distribution over the antenna is as indicated at the top of the diagram. As mentioned above, the radiation resistance for this arrangement is $36 \cdot 6 \ \Omega$. At the right side of the same figure is drawn a similar curve for a flat-top antenna, with a very large capacity at the top and a very small capacity at the bottom. The current distribution is then just reversed

and the radiation resistance is less than in the former case, and
is equal to $32\cdot2\ \Omega$. The dotted lines (for both sides half-
circles) represent the curves that would be obtained using the
ordinary theory, where the phase differences of waves arriving
from different parts of the antenna at a certain point are not
taken into account, in which case the excessive value of $40\ \Omega$
is found.

It is obvious that when there is a current node in the middle
of the antenna the radiation in horizontal directions from one-

$$\lambda = 4l$$
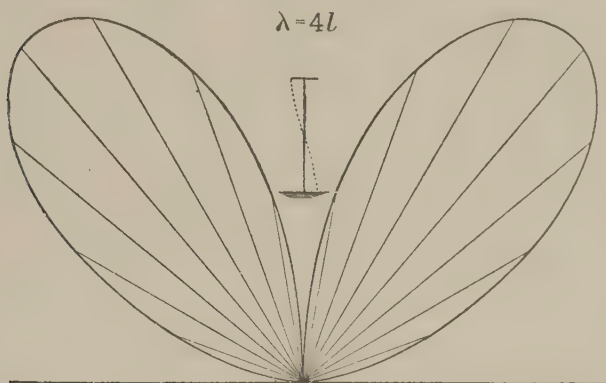
FIG. 5

half of the aerial will, by interference, just annul the radiation
from the other half. According to Zenneck's formula,*

$$R_\rho = 160\pi^2\left(\frac{a}{\lambda}\right)^2,$$

where

$$a = \frac{\dfrac{1}{l/2}\displaystyle\int_0^{l/2} i_x d_x}{T_0},$$

and $\dfrac{l}{2}$ = length of the antenna, there would be in this case no
radiation at all. In fact, there will be a radiation, occasion-
ally even of a great amount, but in more upward directions.

I have calculated some diagrams for different wave-lengths,
but with a current node in the middle of the antenna. Fig. 5
represents this case for $\lambda = 4l$, Fig. 6 for $\lambda = 2l$, Fig. 7 for $\lambda = 4/3l$,
Fig. 8 for $\lambda = l$ and Fig. 9 for $\lambda = \frac{2}{3}l$. Each of these cases, of

* Zenneck, " Lehrb. der Drahtl. Telegr.," 1915.

$\lambda = 2l$

Fig. 6.

$\lambda = {}^{4}/_{3}l$

Fig 7.

$\lambda = l$

Fig. 8.
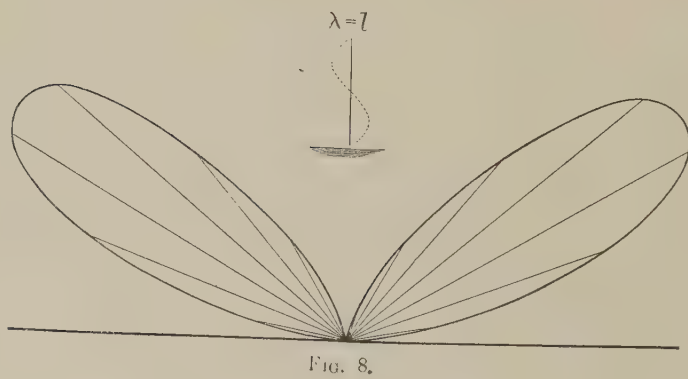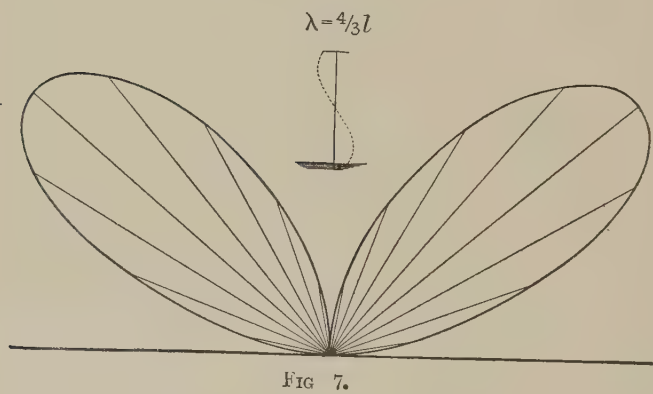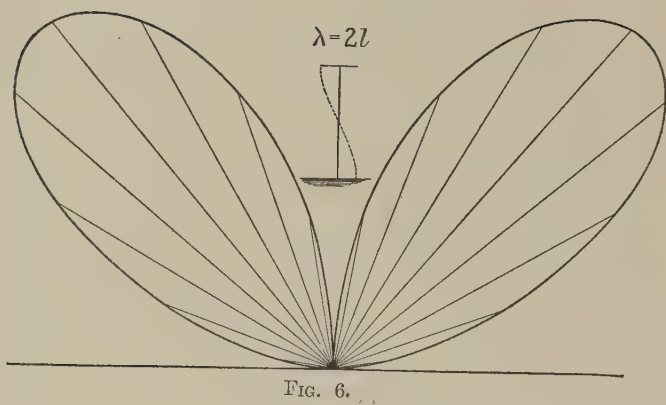
course, demands its own peculiar loading at the top and bottom of the antenna. The diagrams show that decreasing the wave-length has the consequence that the radiation will principally be confined more and more between two cones with a smaller difference of apex angle, so that the characteristic curve

$\lambda = \frac{2}{3}l$
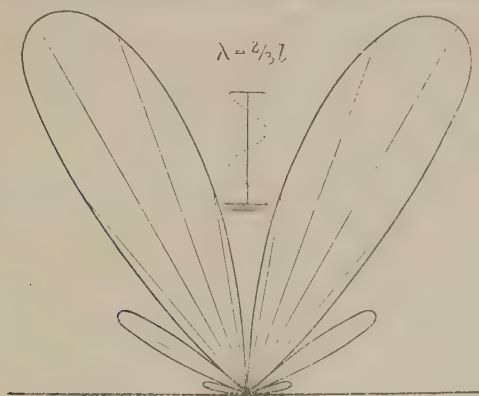
Fig. 9.

becomes more favourable, while at the same time the main part of the radiation is directed in a lower direction. Formula (30) shows that the total amount of radiated energy in each case increases with the decrease of the wave-length. Making

$\lambda = \frac{4}{3}l$
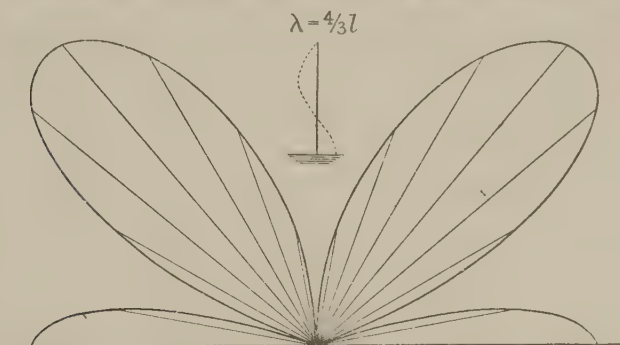
Fig. 10.

the wave-length still shorter (by using a higher harmonic) it appears that the radiation will split up in different parts as shown by Fig. 9, in which $\lambda = \frac{2}{3}l$.

Exciting an antenna in a higher harmonic in such a way that there is no current-node at the middle of it, gives a radia-

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...

tion split up again in several parts, of which one will be in the horizontal direction. This is shown in Fig. 10 for a wavelength $\lambda = 4/3l$ on a vertical rod without any special loading, with the consequent current distribution over the aerial as indicated in the diagram.

It must be mentioned that all curves of the forms given by (32), (33), (34) and (35) are independent of the distance from the antenna, if only that distance is large in comparison with the dimensions of the antenna and the wave-length used, for which case the Poynting vector is a straight line.

The theoretical possibi ity of loading an antenna and exciting it in such a way that no radiation will occur in horizontal directions may experimentally be of value to de:ect the existence of ionised refracting layers in the atmosphere as calculated by Eccles,* for which case the current in a receiving antenna would increase to a certain extent, while reception is made at a greater distance from the sending-station.

Pender Electrical Laboratory,
University College, London.

ABSTRACT.

The Paper consists of a mathematical treatment of the subject, the following being some of the conclusions arrived at : The radiation resistance of a loaded antenna, and also the radiation from the antenna, are dependent not only on the wave-length, but on the current values at the top and bottom. The radiation cannot, therefore, be written $\Sigma = \dfrac{A T^2 l^2}{\lambda^2}$ where A is constant and T is the R.M.S. current at the base, as is done in most textbooks. Rüdenberg's formula for flat-top or umbrella antennæ is valid only for very long wave-lengths, with a capacity at the top of the antenna very large compared with that of the vertical part, and Austin's table of radiation resistances up to ratios of $\dfrac{l}{\lambda} = 0 \cdot 4$ is based on an unjustifiable extrapolation of Rüdenberg's results.

The Paper also treats of the directions in which the energy is most strongly radiated under different conditions.

DISCUSSION.

Prof. J. A. FLEMING communicated the following remarks : The Paper seems to me to be valuable and the analysis correct. The formula for the radiation resistance of a flat-topped antenna, which is commonly called Rüdenberg's formula, has no claim to be attributed to him ; it can be at once derived from Hertz's expressions for the forces due to a small

* THE ELECTRICIAN, September 19, 1913.

oscillator. As I have shown in my book on the " Principles of Electric Wave Telegraphy," third edition, Hertz's reduced expressions are obtained on the conditions that the current has the same value at all points in the rod of the dumb-bell oscillator ; and hence applies also to the reduced formula for radiation resistance. The polar radiation curves which the author has delineated for the higher harmonics of the loaded antenna seem consistent with the delineation of the field of electric force round a rod oscillator given by F. Hack from the equations of Abraham (*see* plate VI., Chapter V., " Principles of Electric Wave Telegraphy "). The possibility of projecting upwards in an inclined direction what is virtually a beam of electric radiation is interesting. By observing where the beam comes to earth it may be possible to determine the height of the reflecting masses of ionised air. There is, however, a certain difficulty in exciting any required high harmonic in an antenna, especially if the capacity and inductance are chiefly located at certain points. When the possibility of experimental work returns it will be interesting to attempt to confirm these results experimentally.

A Demonstration of a Method of Preventing Sparking at a rapid Make and Break, which incidentally produces Colloidal Platinum, was given by Dr. A. GRIFFITHS, *at the Meeting on June 8th,* 1917.

THE apparatus exhibited was one described by him in the " Philosophical Magazine " for March, 1895, p. 232. The device consists of a series of electrolytic cells placed as a shunt across the spark-gap.

The electrodes consist of platinum and the electrolyte of strong sulphuric acid. The cells polarise, and on making the gap an E.M.F. is introduced opposed to the E.M.F. of the battery, so that the current rapidly diminishes, decomposing the liquid and doing chemical work.

Dr. Griffiths made the following statements :—

1. The platinum cathodes disintegrate and a colloidal solution of platinum is formed.

2. The cathode on the negative side of the spark-gap generally disintegrates to the greatest extent ; the next cathode disintegrates less, and so on, the least disintegration occurring in the cell at the positive side of the spark-gap.

3. The cathodes develop, to the naked eye, an appearance as if they were covered with platinum black. Certain plates examined under the microscope seemed covered with numerous craters.

4. The production of gas does not appear to be the same in each of the electrolytic cells in series ; sometimes no gas at all appears to be evolved from the most negative cathode.

5. The rate of disintegration of a cathode appears to be small when the cathode is first placed in the sulphuric acid, and appears to increase to a maximum in course of time.

6. One specimen of platinum appears to behave differently from another.

Dr. Griffiths added that by using electrodes made of gold leaf floated on glass with the aid of methylated spirits, he had obtained what was probably colloidal gold. Prof. Hicks, of Sheffield, had told him that he had used electrolytic cells to stop sparking at a pendulum-mercury intermittent contact. He had not observed disintegration of the platinum as far as he (Dr. Griffiths) was aware.

DISCUSSION.

Dr. R. S. WILLOWS said he was interested in the chemical aspect of the experiment. It was probable that Dr. Griffiths was the first to obtain colloidal platinum, though he did not call it by that name when the result was originally described. The difference in the intensity of the action in different parts of the circuit reminded him of the similar phenomenon which occurs in a train of X-ray valves, in which most of the action takes place in one valve of the train. Measurements of the potential across each cell would be instructive.

Prof. BOYS mentioned that he had used electrolytic cells to quench the sparking of a pendulum contact in his gravitation experiments. He had not noticed any action on the platinum. To what did Dr. Griffiths attribute the effect ?

Mr. A. CAMPBELL communicated the following remarks : Many years ago Helmholtz introduced the use of a condenser, shunted by a suitable resistance, across the spark gap for the purpose of suppressing the spark. More recently a better system was employed by the engineers of the G.P.O., who kindly communicated it to the National Physical Laboratory, where it has been used with the greatest success on all kinds of spark-gaps (from 1 break per second up to 1,000 per second). This consisted in putting the resistance in series with the condenser. By adjusting the value of the resistance to suit the particular circuit it is nearly always possible to suppress the spark. There is no leak across the break, as in the shunt system.

Dr. GRIFFITHS, in reply to Prof. Boys, said that he thought the disintegration was due to the inductive rush of the current, but many points required investigation.

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